

FINDING PLANAR SURFACES IN KNOT- AND LINK-MANIFOLDS

WILLIAM JACO, J. HYAM RUBINSTEIN, AND ERIC SEDGWICK

ABSTRACT. It is shown that given any link-manifold, there is an algorithm to decide if the manifold contains an embedded, essential planar surface; if it does, the algorithm will construct one. The method uses normal surface theory but does not follow the classical approach. Here the proof uses a re-writing method for normal surfaces in a fixed triangulation and may not find the desired solution among the fundamental surfaces. Two major results are obtained under certain boundary conditions. Given a link-manifold M , a component B of ∂M , and a slope γ on B , it is shown that there is an algorithm to decide if there is an embedded punctured-disk in M with boundary γ and punctures in $\partial M \setminus B$; if there is one, the algorithm will construct one. Again, while normal surfaces are used, we may not find a solution among the fundamental surfaces. In this case we use induction on the number of boundary components of the link-manifold. It also is shown that given a link-manifold M , a component B of ∂M , and a meridian slope μ on B , there is an algorithm to decide if there is an embedded punctured-disk with boundary a longitude on B and punctures in $\partial M \setminus B$; if there is one, the algorithm will construct one. This is shown to follow from the previous result using a link-manifold related to M and called the link-manifold obtained from M by Dehn drilling along the slope μ . The properties of minimal vertex triangulations, layered-triangulations, 0-efficient triangulations and especially triangulated Dehn fillings are central to our methods. We also use an average length estimate for boundary curves of embedded normal surfaces; the average length estimate shows, in quite general situations, that given the link-manifold M by a triangulation \mathcal{T} , then all normal surfaces of a bounded genus must have a short boundary curve on some boundary of M . The constant that determines how short is completely determined by the fundamental surfaces in (M, \mathcal{T}) . A version of the average length estimate with boundary conditions also is derived.

1. INTRODUCTION

This work began in the eighties with an attempt to develop a singular normal surface theory as a means toward solving the Word Problem for 3-manifold groups.

Date: February 2, 2008.

1991 *Mathematics Subject Classification.* Primary 57N10, 57M99; Secondary 57M50.

Key words and phrases. word problem, triangulated Dehn filling, knot, link, planar surface, punctured-disk, normal surface, average length estimate, algorithm, 0-efficient, minimal vertex triangulation, layered-triangulation.

The first author was partially supported by NSF Grants DMS9971719, DMS0204707 and DMS0505609, The Grayce B. Kerr Foundation, The American Institute of Mathematics (AIM), and The Lois and Fred Gehring Visitor Chair at University of Michigan.

The second author was partially supported by The Australian Research Council and The Grayce B. Kerr Foundation.

The third author was partially supported by US-Israel BSF2002039 and The Grayce B. Kerr Foundation.

The Word Problem for 3-manifolds can be formulated as a decision problem given a knot-manifold.

A compact, orientable 3-manifold with nonempty boundary, each component of which is a torus, is called a *link-manifold*. If the boundary is connected, then we say it is a *knot-manifold*. We are interested in algorithms to determine if a given knot- or link-manifold contains an interesting planar surface. Often we are interested in how the planar surface sits within the manifold and for this we use some special terminology.

The isotopy class of a non contractible simple closed curve in a torus is called a *slope*. Classically, the study of knot-manifolds has been as the exterior of knots embedded in some other manifold. In such a situation, there is a unique slope on the boundary of the knot-manifold, corresponding to the isotopy class of curves on its boundary that bound a disk in the solid torus neighborhood of the knot. Such a curve is called a meridian. When we are given a knot-manifold along with a slope in a component of its boundary, we use the term *meridian* for the given slope. A slope with geometric intersection *one* with the meridian is called a *longitude*. There are infinitely many longitudes for a meridian, each obtained from any of the others by Dehn twists about the meridian.

If D is a disk, $\{p_1, \dots, p_K\}$ are distinct points in the interior of D , and $\{\eta(p_1), \dots, \eta(p_K)\}$ are pairwise disjoint regular neighborhoods of the points with $\eta(p_i) \subset \overset{\circ}{D}$, $1 \leq i \leq K$, then we say that $P = D \setminus \bigcup_{i=1}^K \overset{\circ}{\eta}(p_i)$ is a *punctured-disk with boundary* $bd(P) = \partial D$ and *punctures* $\partial\eta(p_1), \dots, \partial\eta(p_K)$. Of course, a punctured-disk is a planar surface but a punctured-disk has a distinguished boundary component, and all other boundary components are called punctures.

In this setting, the following is the Word Problem for the fundamental groups of closed 3-manifolds.

WORD PROBLEM (closed 3-manifolds). *Given a knot-manifold M and a meridian on ∂M . Decide if a longitude bounds a (possibly) singular punctured-disk in M with punctures meridians.*

We note that if any longitude bounds a singular punctured-disk, then all longitudes bound a singular punctured-disk.

Our approach was to understand singular punctured-disks by considering them as normal surfaces. One quickly notices that the analogous question for an embedded punctured-disk, which must actually be a disk having no punctures, is the Classical Unknotting Problem. The Unknotting Problem was solved for knot-manifolds in S^3 by W. Haken [3], where he showed that given a knot in S^3 it can be decided if it is the unknot. On the other hand, there is a curious analogy to link-manifolds, which does not have so fortunate an outcome; in fact, it is an insoluble problem.

WORD PROBLEM (finitely presented groups). *Given a link-manifold M , a component B of ∂M , and a meridian slope on B . Decide if there is a (possibly) singular punctured-disk in M with boundary slope a longitude in B and punctures in $\partial M \setminus B$ or meridians on B .*

The Word Problem for finitely presented groups is not solvable; hence, the preceding is an insoluble decision problem for link-manifolds.

In the last section of this paper, we discuss the equivalence of these statements of the Word Problem with other familiar versions.

Being unable to make progress in the case of singular normal surfaces, we decided to investigate if analogous problems for embedded surfaces had solutions and found several interesting questions regarding embedded planar surfaces in knot- and link-manifolds. Unfortunately, at that time, we did not make much progress on these either and laid the problems aside until the late nineties. In the late nineties, we discovered a number of new tools for working with normal surfaces and special triangulations, both of which lend themselves nicely to algorithmic problems. We soon obtained that given a link-manifold it can be decided if there is an embedded, essential planar surface in the manifold. We use the term *essential* in describing a properly embedded surface in a 3-manifold in this work to mean that the surface is incompressible and is not parallel into the boundary. Recall that for link-manifolds, an incompressible surface is also ∂ -incompressible or is an annulus; and if the knot-manifold is irreducible, the annulus must be parallel into the boundary. It was at that time we returned to these problems. We delayed writing these results for publication until now.

In Section 4 we give the following general result about finding interesting planar surfaces.

Theorem. *Given a link-manifold there is an algorithm to decide if it contains a properly embedded, essential, planar surface; if it does, the algorithm will construct one.*

Possibly the most interesting aspects of this result are the tools used and the method of its proof. Typically, algorithms using normal surface theory follow a standard format.

Firstly, there is an existence step where one shows that if a given manifold contains an embedded surface with a property \mathcal{P} , then it contains a normal one with this property \mathcal{P} . However, at some points in this work, we need to modify the triangulation in order to assure the existence of a normal surface with the desirable property.

Secondly, there is a recognition step. The necessary algorithms for recognition that a given normal surface is an essential surface are given in [12]; however, we also use from [12] that if F is an embedded, essential surface and is least weight in its isotopy class, then every normal surface that has its projective representative in the carrier of F also is an embedded, essential, normal surface. The latter result first appeared in [6] using handle-decompositions and later in [1] for closed surfaces and triangulations.

Thirdly, in the classical approach, the big (typically, by far the hardest) step is to show that if there is a surface with a property \mathcal{P} , then there is one among the fundamental surfaces.

We note that a positive solution to the first two steps places us in the common situation for recursively enumerable problems. Namely, given a 3-manifold M by a triangulation \mathcal{T} , all normal surfaces in M (with respect to \mathcal{T}) can be constructed. Thus we merrily go about constructing the normal surfaces. If one with property \mathcal{P} exists, it is normal and if it is normal, we can recognize it. Thus if the given manifold has such a surface, we will eventually find one. However, if there is none, we do not know this and do not know when to stop looking. There are only finitely many fundamental surfaces; hence, if our surface must be among this finite set, which we

can construct straight away, then we have solved the problem. Our algorithms do not have the classical step three; in fact, we might need to go quite far afield of the fundamental solutions of the given triangulation but we do find a finite set in which to look and all surfaces in this finite set are normal in our given triangulation. In Section 4, the proof is by a method we call a *re-writing process*. A re-writing process was also used in [9]; the re-writing process here is different but is the same principle.

In Section 5, we answer the problem for embedded punctured-disks analogous to the singular problem above for the Word Problem for finitely presented groups. It is given in the second of the next two theorems.

Theorem. *Given a link-manifold M , a component B of ∂M , and a slope γ in B , there is an algorithm to decide if M contains an embedded punctured-disk with boundary having slope γ and punctures in $\partial M \setminus B$. If there is one, the algorithm will construct one.*

Theorem. *Given a link-manifold M , a component B of ∂M , and a meridian in B , there is an algorithm to decide if M contains an embedded punctured-disk with boundary having slope of a longitude in B and punctures in $\partial M \setminus B$. If there is one, the algorithm will construct one.*

Again, we do not conform to the classical third step of finding our solution among the fundamental surfaces. Moreover, our proof uses an interesting method for link-manifolds; we use induction on the number of boundary components. For rather subtle reasons, we can not use induction in Section 4.

Besides not using the classical form of proof for the above problems, we also call upon a number of new results on triangulations and new tools in normal surface theory. We discuss what we need from the literature and provide results necessary for this work in Section 3 and later sections. For example, our methods typically require minimal-vertex triangulations or at least triangulations that have at most one vertex in each boundary component. In some situations we need 0-efficient triangulations, which are minimal vertex triangulations for knot- and link-manifolds. These triangulations are quite general and given a 3-manifold via any triangulation, there are algorithms that modify the given triangulation to one of these that fits into our methods. In particular, in Section 4, we generalize a result from [9] and prove the following prime decomposition theorem, where $n(Q)$ in the connected sum decomposition means the connected sum of n copies of the manifold Q and $\text{Card}(\mathcal{T})$ for a triangulation \mathcal{T} stands for the number of tetrahedra of \mathcal{T} .

Theorem. *Given a link-manifold M via a triangulation \mathcal{T} , there is an algorithm to construct a prime decomposition*

$$M = p(S^2 \times S^1) \# q(\mathbb{R}P^3) \# r(D^2 \times S^1) \# M_1 \# \cdots \# M_n,$$

where p, q and r are nonnegative integers and each M_i is given by a 0-efficient triangulation \mathcal{T}_i , $i = 1, \dots, n$, respectively; furthermore, $\sum_{i=1}^n \text{Card}(\mathcal{T}_i) \leq \text{Card}(\mathcal{T})$.

We also use results from [14] and [11] on layered-triangulations of the solid torus and the classification of normal surfaces in minimal layered triangulations of the solid torus. This is used in conjunction with triangulated Dehn fillings, which were introduced in these same two references. If we are given a link-manifold M with a triangulation \mathcal{T} , having just one vertex in each boundary component, then for slopes $\alpha_1, \dots, \alpha_K$ on distinct boundary components of M , there is a natural way to

triangulate the Dehn filling $M(\alpha_1, \dots, \alpha_K)$ with a triangulation that is \mathcal{T} on M and is a minimal layered-triangulation of each of the solid tori added. We discuss this and some of the results we use from the literature in Section 3.

We assume the reader is familiar with the basic concepts from normal surface theory; however, in Section 3, we identify some of the particular concepts and results that we use. Among these, one which may not be so familiar, is that mentioned above where if F is an embedded, essential normal surface and is least weight in its isotopy class, then every normal surface that projects into the carrier of F is embedded and essential. We also define and use the length of the boundary of a normal surface. Another major tool related to the boundary of a normal surface, also used in [11], is what we call the *Average Length Estimate* (ALE) for the boundary of a normal surface. For a manifold M with triangulation \mathcal{T} , there is a constant C dependent only on M and \mathcal{T} so that every normal surface of bounded genus has the average length of its boundary no larger than C ; hence, in particular, for link-manifolds (under the right conditions) a normal planar surfaces must have a short boundary slope on some boundary of the link-manifold. This enables us to find a finite family of (short) slopes in which to do Dehn fillings and apply an induction hypothesis. In using triangulated Dehn fillings in conjunction with ALE, we are able to have very strong control on the complexity of our methods.

Among the questions we consider is one to determine if there is a planar normal surface with a prescribed boundary slope on some boundary component. This and other questions that involve the boundary of the normal surface are called *boundary conditions*. Section 5 considers a theory of normal surfaces with boundary conditions. In particular, we show that for the triangulations we use, given a slope on the boundary, there is a set of matching equations so that we get a normal solution space having only normal surfaces that meet the given boundary component in the given slope. We may have closed normal surfaces and surfaces that meet other boundary components in a totally uncontrolled way. These method can be greatly generalized; we did not do that here. We also adapt ALE to normal surfaces with boundary conditions in Section 5.

In the last section we return to a discussion of those versions of the Word Problem given above and what we might call the classical versions. Following the solution of the Geometrization Conjecture, we know that the Word Problem for 3-manifold groups is solvable; however, we remain curious as to the existence of a straight forward method, say, in the spirit of the solution to the Word Problem for the fundamental groups of Haken manifolds, given by F. Waldhausen [19].

2. BACKGROUND MATERIAL

We shall assume the reader has familiarity with our notion of triangulations, as well as a basic knowledge of normal surface theory. The references [9, 11, 14] serve as good background material for both triangulations from our point of view and basic facts on normal surfaces. We have, however, collected some facts and background in this section, which are particularly relevant to this work.

2.1. Triangulations. One of the interesting aspects of this work is not only the effectiveness of minimal vertex triangulations in understanding the combinatorial and algorithmic problems we encounter but also in enabling us to understand the topology better. The following is typical of the type of triangulations we like.

2.1. Theorem. [11, 13] *Suppose M is a compact, orientable 3-manifold with boundary, no component of which is a 2-sphere. Then any triangulation of M can be modified to a triangulation having all vertices in ∂M and just one vertex in each component of ∂M .*

A proof is given in both the cited references. The idea is quite straight forward. First, there is an algorithm due to R.H. Bing [2] that can be used to modify a given triangulation to one having all vertices on the boundary. This step was not mentioned but is necessary in the proof given in [11]. Having all vertices in the boundary, then the method of “closing-the-book”, described in detail in Theorem 3.3 of [11], finishes the proof.

A manifold having a triangulation with just one vertex (closed or possibly compact and bounded with just one boundary component) or a triangulation with all vertices in the boundary and just one vertex in each boundary component (compact with boundary) can not have a triangulation with fewer vertices. We shall refer to such triangulations as *minimal-vertex triangulations*. This is different from a minimal triangulation of the manifold M ; a triangulation \mathcal{T} of M is a *minimal triangulation* if and only if for any triangulation \mathcal{T}' of M , $\text{Card}(\mathcal{T}) \leq \text{Card}(\mathcal{T}')$, where $\text{Card}(\mathcal{T})$ is used to denote the number of tetrahedra of a triangulation. For irreducible 3-manifolds, distinct from S^3 , $\mathbb{R}P^3$, $L(3, 1)$, and \mathbb{B}^3 , minimal triangulations are minimal-vertex triangulations. We suspect minimal triangulations are also minimal-vertex triangulations for reducible 3-manifolds but have not established this. Minimal-vertex triangulations are completely general; the above theorem shows that any compact 3-manifold with boundary (no component of which is a 2-sphere) admits a minimal-vertex triangulation and a proof is given in [14] that any closed 3-manifold admits a one-vertex triangulation.

2.2. Normal surfaces. We collect here the main results from normal surface theory we will need along with our conventions for notation. Primary sources for this material are [7, 8, 11, 9, 12].

If M is a 3-manifold, a triangulation of M selects a family of surfaces called normal surfaces. Typically a normal surface is defined as an embedded surface that meets the tetrahedra of the triangulation in normal triangles and normal quadrilaterals. An isotopy of M that is invariant on the various simplices of the triangulation is called a *normal isotopy*. A normal isotopy class of normal triangles or of normal quadrilaterals is called a *triangle type* or *quad type*, respectively. We caution the reader that with the triangulations we are using, where the simplices are only embedded on their interiors and may have identifications on their boundaries, an embedded surface is normal if and only if its pull back to the tetrahedra before face identifications is a collection of normal triangles and normal quadrilaterals.

We shall use standard normal coordinates. In this case, there are four normal triangle types and three normal quad types for each tetrahedron, giving each normal surface a parametrization with $7t$ variables, where t is the number of tetrahedra in the triangulation. Hence, an embedded normal surface determines a unique nonnegative integer lattice point in \mathbb{R}^{7t} . The triangulation also determines a system of homogeneous linear equations, the *matching equations*; its solution space meets the nonnegative orthant of \mathbb{R}^{7t} in a cone called the *solution cone*, which we denote $\mathcal{S}(M, \mathcal{T})$. We say two normal surfaces satisfy the same *quadrilateral condition* if they do not meet any one tetrahedron in distinct quadrilateral types; equivalently, there is an additional set of conditions placed on the coordinates in the solution

cone where for each tetrahedron two of the quadrilateral types have been set to zero. This algebraic condition is also called a quadrilateral condition. There are 3^t possible quadrilateral conditions. Each integer lattice point in the solution cone corresponds to a (possibly singular) normal surface. This correspondence is one-one between embedded normal surfaces in M with respect to \mathcal{T} and the integer solutions in the solution cone that also satisfy a quadrilateral condition. Those points in the solution cone that have norm one in the ℓ_1 -norm ($\sum x_i = 1, x_i \geq 0$) form a compact, convex, linear cell called the *projective solution space* for M with triangulation \mathcal{T} , which is denoted $\mathcal{P}(M, \mathcal{T})$. The solution cone is the cone over $\mathcal{P}(M, \mathcal{T})$ with vertex the origin. Each point in the solution cone has a unique projection into projective solution space. If two points in the solution cone project to the same point in $\mathcal{P}(M, \mathcal{T})$, we say they are *projectively equivalent*. If F is a point in the solution space and \overline{F} is its projection in $\mathcal{P}(M, \mathcal{T})$, then we call the minimal dimensional closed face of $\mathcal{P}(M, \mathcal{T})$ containing \overline{F} the *carrier of F* and denote it by $\mathcal{C}(F)$. We will not distinguish notation between an embedded normal surface F and its parametrization F in \mathbb{R}^{7t} .

From the Hilbert Basis Theorem, there is a *unique, minimal* finite set of integer lattice solutions in $\mathcal{S}(M, \mathcal{T})$, F_1, \dots, F_K , so that for F any integer lattice point in $\mathcal{S}(M, \mathcal{T})$, we have

$$F = \sum n_i F_i,$$

where n_i is a nonnegative integer. We call such a family F_1, \dots, F_K *fundamental solutions*. If V is an integer lattice point in $\mathcal{S}(M, \mathcal{T})$, V projects to a vertex \overline{V} of $\mathcal{P}(M, \mathcal{T})$, and if for any integer lattice point V' that projects to \overline{V} we have $V' = kV$ for some positive integer k , we call V a *vertex solution*. All vertex solutions must be among any set of fundamental solutions. The vertex solutions may be found using any one of a number of methods from linear programming and the fundamental solutions may be found once one has the vertex solutions. We have the following characterizations of fundamental and vertex solutions.

- (1) F is a fundamental solution if and only if whenever A and B are solutions and $F = A + B$, either $A = 0$ or $B = 0$.
- (2) V is a vertex solution if and only if whenever A and B are solutions and there is a positive integer k so that $kV = A + B$, then $A = k'V$ and $B = k''V$, k', k'' positive integers.

If F and F' are normal surfaces and they satisfy the same quadrilateral conditions, then using standard cut-and-paste techniques from 3-manifold topology, there is a unique way to form a normal surface from F and F' called the *geometric sum* of F and F' . Since the geometric sum of two embedded normal surfaces F and F' is parameterized by the coordinate sum of their parameterizations, we also write the geometric sum of F and F' as $F + F'$.

There are several forms of complexity associated with normal surfaces; two of the simplest ones are its weight, analogous to area, and the length of its boundary. If F is a normal surface in M , then F is in general position with respect to the 2-skeleton of the triangulation \mathcal{T} ; we define $wt(F) = \text{Card}(F \cap \mathcal{T}^{(1)})$ to be the *weight of F* , where $\text{Card}(S)$ is the cardinality of S . Similarly, we define $L(\partial F) = \text{Card}(\partial F \cap \mathcal{T}^{(1)})$ as the *length of ∂F* .

There is a special form for geometric addition; namely, the geometric sum $F + G$ is said to be in *reduced-form* if for all possible ways to write $F + G$ the number

of components of $F \cap G$ is minimal; i.e., if $F + G = F' + G'$, then the number of components of $F \cap G$ is no larger than the number of components of $F' \cap G'$. We have the following very useful observations, which we learned from [16]. Clearly, any geometric sum may be written in reduced-form.

2.2. Lemma. *Suppose the geometric sum $F + G$ is defined and in reduced-form. Then*

- *a component of $F \cap G$ does not separate both F and G ;*
- *if $F + G$ is connected, then both F and G are connected.*

We note the following easily established facts where F and G are embedded normal surfaces that satisfy the same quadrilateral conditions.

- (1) $\chi(F + G) = \chi(F) + \chi(G)$,
- (2) $wt(F + G) = wt(F) + wt(G)$, and
- (3) $L(\partial(F + G)) = L(\partial F) + L(\partial G)$.

2.3. Other Triangulations. There are some special triangulations we will be using and we briefly discuss these triangulations and related useful facts.

- **0-efficient triangulations.** A triangulation of a closed 3-manifold is said to be 0-efficient if and only if the only normal 2-spheres are vertex-linking; if the manifold has boundary, a triangulation is said to be 0-efficient if and only if the only normal disks are vertex-linking. From [9] we have the following about 0-efficient triangulations.

Suppose \mathcal{T} is a 0-efficient triangulation of the 3-manifold M .

If M is closed, then

- (i) M is irreducible and contains no embedded $\mathbb{R}P^2$.
- (ii) \mathcal{T} has one vertex or $M = S^3$; if $M = S^3$, then \mathcal{T} has at most two vertices.

If M has nonempty boundary, then

- (iii) \mathcal{T} has no normal 2-spheres,
- (iv) M is irreducible and ∂ -irreducible.
- (v) All the vertices of \mathcal{T} are in ∂M and there is just one vertex in each boundary component or M is a 3-cell.

Hence, a 0-efficient triangulation is a minimal-vertex-triangulation, except possibly for S^3 and the 3-cell. For a 3-cell with a 0-efficient triangulation, then the triangulation is expected to have precisely three vertices, all in the boundary; it is easy to see all vertices must be in the boundary but we have not been able to show there are only three.

In Theorem 4.6, we show that given a link-manifold M , we can construct a prime decomposition of M , where the irreducible and ∂ -irreducible factors have 0-efficient triangulations. It is shown in [9] that any compact, irreducible, ∂ -irreducible, orientable 3-manifold, distinct from $\mathbb{R}P^3$, admits a 0-efficient triangulation. In particular, minimal triangulations of these manifolds are 0-efficient.

- **One-vertex triangulations and slopes in tori.** Up to homeomorphism of the torus there is a unique one-vertex triangulation. It has two triangles, three edges and (of course) one vertex. For any triangulation of a surface an essential (not contractible) simple closed curve is isotopic to a normal curve; however, for a one-vertex triangulation of a torus there is more. Namely, by Lemma 3.5 [11] essential

curves in a one-vertex triangulation of a torus are isotopic if and only if they are normally isotopic; thus in such a triangulation there is a unique normal isotopy class for each essential simple closed curve. By Lemma 3.4 [11], we also note that in any one-vertex triangulation of a closed surface, the only trivial (contractible) normal curve is vertex-linking. Hence, in a one-vertex triangulation of a torus, slopes and normal isotopy classes of essential simple closed curves are in one-one correspondence.

In a one-vertex triangulation of a torus, we say two slopes are *complementary* if the geometric sum of their normal representatives is a union of trivial (vertex-linking) curves. Each slope has a unique complementary slope. The following (Proposition 3.7 of [11]) is a fundamental result about slopes of boundaries of *normal* surfaces in 3-manifolds having tori in their boundary and triangulations that induce one-vertex triangulations on these boundary tori.

2.3. Theorem. [11] *Let M be an orientable 3-manifold having a component of its boundary a torus, T , and let \mathcal{T} be a triangulation of M that restricts to a one-vertex triangulation of T . Suppose S_1 and S_2 are embedded normal or almost normal surfaces and $\partial S_1 \subset T$. If S_1 and S_2 satisfy the same quadrilateral conditions and both meet T in non-trivial slopes, then these slopes are either equal or complementary.*

Some components of the boundaries of S_1 and S_2 may be trivial curves in the boundary of M ; however, it is implicit in the theorem that there is an essential curve from each of S_1 and S_2 in T to determine slopes. This theorem gives the result that for a knot-manifold with a triangulation inducing a one-vertex triangulation on the boundary torus, there are only finitely many boundary slopes for normal and almost normal surfaces. In particular, it gives the result, discovered earlier by A. Hatcher [4], that for a knot-manifold M there are a finite number of slopes bounding embedded, incompressible and ∂ -incompressible surfaces in M .

If we choose two slopes, having isotopy classes λ and μ on a torus and geometric intersection one, then they determine a basis for the first homology of the torus. Representing their homology classes also by λ and μ , respectively, we have that any slope α can be represented as $\alpha = a\lambda + b\mu$, where a and b are relatively prime integers (possibly $a = 0, b = \pm 1$ or $a = \pm 1, b = 0$). We define the *distance* between the slopes α and β , denoted $\langle \alpha, \beta \rangle$, to be their geometric intersection number; hence, for a basis λ, μ and $\alpha = a\lambda + b\mu, \beta = c\lambda + d\mu$, we have $\langle \alpha, \beta \rangle = |ad - bc|$. $\langle \cdot, \cdot \rangle$ is not a true distance function but we do have $\langle \alpha, \beta \rangle = 0$ if and only if $\alpha = \beta$ and $\langle \alpha, \beta + \gamma \rangle = \langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle$ and $\langle \alpha, m\beta \rangle = m\langle \alpha, \beta \rangle$. Once we have designated a basis for the homology of the torus, slopes are in one-one correspondence with $\mathbb{Q} \cup \infty, \mathbb{Q}$ the rationals.

- Layered-triangulations of the solid torus and triangulated Dehn fillings.

Layered-triangulations of the solid torus are studied extensively in [14]. Both layered triangulations of the solid torus and triangulated Dehn fillings are used in [10, 11, 14]. We provide a brief review here.

Suppose M is a compact 3-manifold with nonempty boundary, \mathcal{T} is a triangulation of M and \mathcal{T}_∂ is the induced triangulation on ∂M . Furthermore, suppose e is an edge in \mathcal{T}_∂ and there are two distinct triangles σ and β in \mathcal{T}_∂ meeting along the edge e . Let $\tilde{\Delta}$ be a tetrahedron distinct from the tetrahedra in \mathcal{T} and let \tilde{e} be an edge in $\tilde{\Delta}$. Suppose $\tilde{\sigma}$ and $\tilde{\beta}$ are the faces of $\tilde{\Delta}$ that meet along \tilde{e} . We can identify \tilde{e} with e and extend this to face identifications from $\tilde{\sigma} \rightarrow \sigma$ and $\tilde{\beta} \rightarrow \beta$, getting a

3-manifold M' homeomorphic with M and a triangulation \mathcal{T}' of M' , having one more tetrahedron than \mathcal{T} . We write $M' = M \cup_e \Delta$ and $\mathcal{T}' = \mathcal{T} \cup_e \tilde{\Delta}$, where Δ is the image of $\tilde{\Delta}$ and say that \mathcal{T}' is obtained from \mathcal{T} by *layering (a tetrahedron) on \mathcal{T} along the edge e* . Notice that this operation transforms the triangulation \mathcal{T}_∂ to the triangulation \mathcal{T}'_∂ by what is called a Pachner or bi-stellar move of type $2 \leftrightarrow 2$ on \mathcal{T}_∂ along the edge e (also called a “diagonal flip” within the quadrilateral $\sigma \cup \beta$).

There is another form of layering, which can be thought of as a degenerate form of what we have just described. For example, there are three ways to layer the back two faces of the tetrahedron $\tilde{\Delta}$ onto the one-triangle Möbius band. See Figure 1.

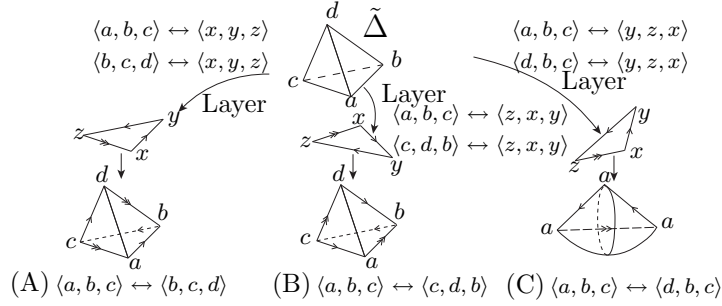


FIGURE 1. One-tetrahedron solid torus and creased 3-cell (layering of a tetrahedron on a one-triangle Möbius band).

In parts (A) and (B) the tetrahedron is layered along the interior (orientation reversing edge) on the one-triangle Möbius band, the labels and arrows give the identifications. Combinatorially, these triangulations are the same. This triangulation of the solid torus will be referred to as *the one-tetrahedron solid torus*. In the last case, Figure 1, Part C, we show a *creased 3-cell* obtained by a single layering of a tetrahedron along the boundary edge of the one-triangle Möbius band; again the labels and arrows give the identification. The Möbius band and the creased 3-cell both have the homotopy type of a solid torus; we think of each as a degenerate layered-triangulation of the solid torus.

With the above notion of layering on a triangulation and starting with the one-triangle Möbius band, we inductively define a triangulation \mathcal{T}_t of the solid torus to be a *layered-triangulation of the solid torus with t -layers* if

- (0) \mathcal{T}_0 is the one-triangle Möbius band,
- (1) \mathcal{T}_1 is either the one-tetrahedron solid torus or the creased 3-cell (both obtained by layering on the one-triangle Möbius band), and
- (t) $\mathcal{T}_t = \mathcal{T}_{t-1} \cup_e \tilde{\Delta}_t$ is a layering along the edge e of a layered-triangulation \mathcal{T}_{t-1} of the solid torus having $t - 1$ layers, $t \geq 1$. See Figure 2.

Note that a layered-triangulation of the solid torus with t layers has t tetrahedra, $2t + 1$ faces with two faces in the boundary, $t + 2$ edges with three edges in the boundary and one vertex, which is in the boundary. It is possible by layering on the creased 3-cell that one does not get a solid torus but a homotopy solid torus; we do not use such layerings.

While there is a unique one-vertex triangulation of the solid torus, there are infinitely many ways, up to homeomorphism of the solid torus, to place a one-vertex triangulation of the torus onto the boundary of the solid torus. Two one-vertex

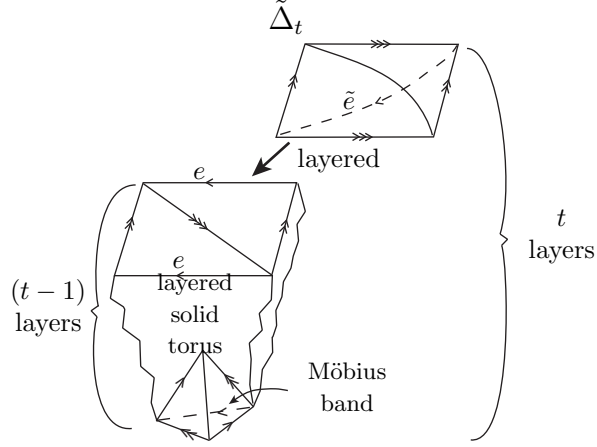


FIGURE 2. Layered-triangulation of a solid torus.

triangulations, \mathcal{T}_∂ and \mathcal{T}'_∂ , on the boundary of the solid torus are equivalent if and only if there is a homeomorphism of the solid torus taking \mathcal{T}_∂ to \mathcal{T}'_∂ . In general, if \mathcal{T}_∂ is a triangulation on the boundary of a 3-manifold M , a triangulation \mathcal{T} of M is an extension of \mathcal{T}_∂ if \mathcal{T} restricted to ∂M is \mathcal{T}_∂ and all the vertices of \mathcal{T} are in ∂M (no vertices are added). We have the following theorem from [14], a version also appears in [11].

2.4. Theorem. *Suppose \mathcal{T}_∂ is a one-vertex triangulation on the boundary of the solid torus. Then \mathcal{T}_∂ can be extended to a layered-triangulation of the solid torus; in fact, there is a unique extension of \mathcal{T}_∂ to a minimal layered-triangulation of the solid torus.*

Here we use *minimal* with layered-triangulation to mean that of all layered-triangulations that extend the triangulation \mathcal{T}_∂ , there is a unique one with the fewest number of tetrahedra. We distinguish the unique minimal layered-triangulation of the solid torus that extends the triangulation \mathcal{T}_∂ on its boundary by saying it is a \mathcal{T}_∂ -layered-triangulation of the solid torus. It is the preferred extension of the equivalence class of \mathcal{T}_∂ on the boundary of the solid torus to a layered-triangulation of the solid torus. We do not know if the \mathcal{T}_∂ -layered-triangulation is the minimal triangulation of the solid torus extending \mathcal{T}_∂ . We conjecture that it is the minimal extension of \mathcal{T}_∂ .

In [14] the normal surfaces in a minimal layered-triangulations of the solid torus are characterized. In this paper we use that for any layered-triangulation of the solid torus, there is a unique normal meridional disk and that in a minimal layered-triangulation of a solid torus the only normal surface with boundary slope meridional is the meridional disk. There are no closed normal or almost normal surfaces in a layered-triangulation of the solid torus.

Layered-triangulations of the solid torus are quite useful for the construction of nice triangulations of Dehn fillings of knot- and link-manifolds.

Suppose M is a link-manifold and \mathcal{T} is a triangulation of M that induces a one-vertex triangulation on each (torus) boundary component. Suppose α is a slope in the component B of ∂M . Let $M(\alpha)$ denote the Dehn filling of M along the slope

α and $\mathbb{T}(\alpha)$ denote the solid torus of the Dehn filling. The triangulation \mathcal{T} induces a triangulation \mathcal{T}_B on $B = M \cap \partial\mathbb{T}$. Hence, by Theorem 2.4, we can extend \mathcal{T}_B to the \mathcal{T}_B -layered-triangulation of the solid torus \mathbb{T} , giving a triangulation $\mathcal{T}(\alpha)$ of the Dehn filling $M(\alpha)$. We call the pair $(M(\alpha), \mathcal{T}(\alpha))$ a *triangulated Dehn filling*; sometimes the triangulated Dehn filling is understood by using just $M(\alpha)$ or $\mathcal{T}(\alpha)$. See Figure 3.

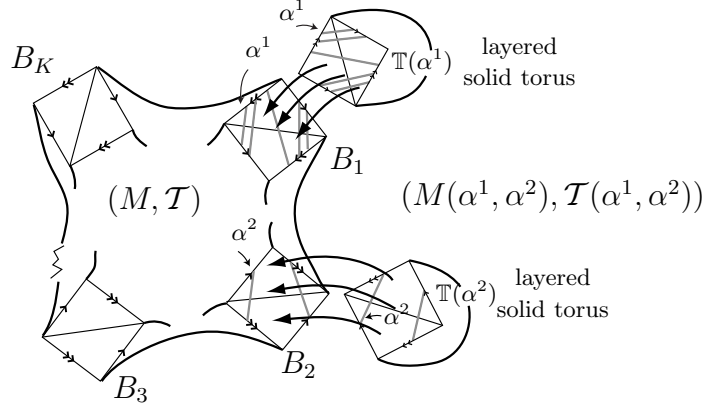


FIGURE 3. A link-manifold with triangulation \mathcal{T} , having precisely one vertex on each boundary component, along with layered-triangulations of the solid torus $\mathbb{T}(\alpha^1)$ and $\mathbb{T}(\alpha^2)$ attached simplicially to the boundary components B_1 and B_2 . This gives a triangulated Dehn filling $\mathcal{T}(\alpha^1, \alpha^2)$ of the Dehn filling $M(\alpha^1, \alpha^2)$.

If M has more than one boundary component, then $M(\alpha)$ is a link-manifold and we can consider Dehn filling it along a slope in its boundary. Hence, if we do Dehn fillings of M along slopes $\alpha^1, \dots, \alpha^k$ in distinct boundary components, to arrive at the Dehn filled 3-manifold $M(\alpha^1, \dots, \alpha^k) = M(\alpha^1, \dots, \alpha^{(k-1)})(\alpha^k)$, we can extend \mathcal{T} using minimal layered-triangulations of the solid torus to get a triangulated Dehn filling with triangulation $\mathcal{T}(\alpha^1, \dots, \alpha^k) = \mathcal{T}(\alpha^1, \dots, \alpha^{(k-1)})(\alpha^k)$.

Notice that for triangulated Dehn fillings of link-manifolds, the triangulation is \mathcal{T} on M and is a minimal \mathcal{T}_{B_i} -layered-triangulation on each solid torus $\mathbb{T}(\alpha^i)$, where B_i is the torus boundary component containing the slope α^i and \mathcal{T}_{B_i} is the triangulation on B_i induced by \mathcal{T} .

Now, if \widehat{G} is a normal surface in $M(\alpha^1, \dots, \alpha^k)$ with a triangulated Dehn filling $\mathcal{T}(\alpha^1, \dots, \alpha^k)$, then \widehat{G} meets M and each of the solid tori $\mathbb{T}(\alpha^i)$ in normal surfaces. This is an aspect of these triangulations, along with the characterization of normal surfaces in a minimal layered-triangulation of a solid torus, that is helpful in understanding normal and almost normal surfaces in Dehn fillings. If the components of \widehat{G} in each of the solid tori $\mathbb{T}(\alpha^i)$ are disks and G is the normal surface in which \widehat{G} meets M , we say G “caps off” and write $G(\alpha^1, \dots, \alpha^k)$ for \widehat{G} . Of course, there are normal surfaces in the triangulated Dehn filling $(M(\alpha^1, \dots, \alpha^k), \mathcal{T}(\alpha^1, \dots, \alpha^k))$ that meet the solid tori in normal surfaces other than the meridional disks; but the “capped-off” normal surfaces are special. We have the following.

2.5. Lemma. *Suppose $(M(\alpha^1, \dots, \alpha^k), \mathcal{T}(\alpha^1, \dots, \alpha^k))$ is a triangulated Dehn filling. If the “capped-off” normal surface $P(\alpha^1, \dots, \alpha^k) = \sum n_q \widehat{G}_q$ is a geometric sum*

of normal surfaces in $(M(\alpha^1, \dots, \alpha^k), \mathcal{T}(\alpha^1, \dots, \alpha^k))$, then $\widehat{G}_q = G_q(\alpha^1, \dots, \alpha^k)$ is a “capped off” surface for each $q, 1 \leq q \leq k$, and $P = \sum n_q G_q$.

Proof. If $\mathbb{T}(\alpha^i)$ is one of the layered solid tori in the triangulated Dehn filling, then $P(\alpha^1, \dots, \alpha^k)$ meets $\mathbb{T}(\alpha^i)$ in a family of meridional disks; furthermore, the components of intersection of each \widehat{G}_q with $\mathbb{T}(\alpha^i)$ are normal surfaces and have geometric sum the collection of meridional disks in which $P(\alpha^1, \dots, \alpha^k)$ meets $\mathbb{T}(\alpha^i)$. It follows from Theorem 2.3 that the components of each \widehat{G}_q meet $\partial\mathbb{T}(\alpha^i)$ in the meridional slope α^i . However, by the classification of normal surfaces in a minimal layered-triangulation of a solid torus, the only such normal surface in $\mathbb{T}(\alpha^i)$ is the unique normal meridional disk. Thus each \widehat{G}_q “caps off” and if $\widehat{G}_q = G_q(\alpha^1, \dots, \alpha^k)$, then we have $P = \sum n_q G_q$. \square

In the situation above, we say we can *re-write* P as the sum $\sum n_q G_q$. This is particularly significant when we have P written as a sum $P = \sum m_j F_j$, where the F_j are fundamental in (M, \mathcal{T}) . Then we re-write $P = \sum n_q G_q$ where $G_q(\alpha^1, \dots, \alpha^k)$ is fundamental in $(M(\alpha^1, \dots, \alpha^k), \mathcal{T}(\alpha^1, \dots, \alpha^k))$. In re-writing P in this way, we have each of the summands G_q meeting the boundary component with slope α^i in the slope α^i , whereas, there is no way of knowing how the various F_j meet the boundary component containing the slope α^i in a link-manifold with multiple boundary components.

2.4. Basic algorithms. In this subsection, we organize some of the algorithms we will be using. Note that we often state the existence of an algorithm about surfaces in a 3-manifold by stating that there is an algorithm that will decide; and if the answer is yes, then the algorithm will construct the desired surface. In many case, however, this is done by showing that the answer is yes if and only if there is a *fundamental surface* that is of the type we seek. And, of course, fundamental surfaces can be constructed. An interesting aspect of the main results of this paper is that the algorithms we develop later do not necessarily find an answer among the fundamental surfaces; to get an answer we have to look, in some cases, rather far afield. We shall, generally, state the conclusions of our results in the stronger terms of finding solutions among the fundamental surfaces, if, indeed, that is where a solution can be found. Many of the results we use here are improved in [12] and [7], showing that desired solutions are already at the vertices of projective solution space if they exist at all. Also, in some cases, a conclusion can be made for any triangulation; in other cases, we must first modify the given triangulation to a triangulation more suitable to a solution of the problem.

The first result we give is generally attributed to W. Haken [3]. We note that the proof attributed to Haken is for a manifold given via a handle decomposition where it is known that the manifold is orientable and irreducible. Such a proof, learned from the methods of [16], is given in [6]. In G. Hemion’s book, [5], an argument is given in the case of triangulations; however, it parallels the argument for handle decompositions, leaving it with a gap. The gap is eliminated in [7] where the following result is proved. We remark that the argument in this generality does involve many details; we can obtain the result much more easily by first constructing a prime decomposition of the manifold from which we will find an essential disk should one exist (see Theorem 4.1 below).

2.6. Lemma. *Suppose M is a compact 3-manifold and B is a component of ∂M . M contains a properly embedded, essential disk with boundary in B if and only if for any triangulation \mathcal{T} of M , there is an essential, normal disk with boundary in B among the fundamental solutions for (M, \mathcal{T}) .*

If we are seeking to know the existence of essential surfaces, then if they exist, they will either be part of a prime decomposition (spheres, projective planes) or exist only if they are in prime factors. In fact, later we reduce our main problem to a problem where it is known that the given manifold is irreducible and ∂ -irreducible. However, in the steps of an algorithm, it often is necessary to alter the given knot or link-manifold to one obtained by a Dehn filling of the given manifold. It is well known that upon Dehn filling one might lose some of the nice features of the original given manifold, such as irreducibility and ∂ -irreducibility.

A version of the following lemma appears within the proof of Lemma 5.11 of [11]; however, that version assumes it is known that the given manifold is irreducible and ∂ -irreducible.

2.7. Lemma. *Suppose M is a compact 3-manifold with nonempty boundary. M contains a properly embedded, essential annulus having its boundary in distinct boundary components B and B' of M if and only if for any triangulation \mathcal{T} of M , there is an embedded, essential, fundamental normal annulus having its boundary components in B and B' .*

Remark. Suppose M is link-manifold and B is a component of ∂M . If there are properly embedded, essential annuli A and A' in M each having a boundary component in B , then either the component(s) of the boundary of A and A' in B have the same slope or B is in the boundary of a prime factor of M that is an I -bundle; i.e., $S^1 \times S^1 \times I$ or the twisted I -bundle over the Klein bottle.

A general version of an algorithm to decide if a given normal surface is an essential surface appears in [12]; the following version follows from early work of Haken [3] with some of the material from [12] to determine if a normal surface in a 3-manifold M is parallel into ∂M . Recall in a link-manifold M a properly embedded surface is essential if and only if it is incompressible and is not an annulus or torus parallel into ∂M . One does not need separately to check ∂ -irreducibility.

2.8. Theorem. *Given a link-manifold M and an embedded, normal surface F in M , there is an algorithm to decide if F is an essential surface in M .*

We also have another very useful result that we can use to conclude that a surface is essential. It was first established in [6] but using handle-decompositions; later it was redone for triangulations in [1] and in the form we use in [12].

2.9. Theorem. *Suppose M is an irreducible, ∂ -irreducible 3-manifold and \mathcal{T} is a triangulation of M . If F is an embedded, essential normal surface in M and is least weight in its isotopy class, then every normal surface with projective class in the carrier of F , $\mathcal{C}(F)$, is embedded and essential in M .*

Later, Tollefson [18] showed that all normal surfaces with projective class in $\mathcal{C}(F)$ are also least weight in their isotopy class.

Finally, given a normal surface, there are various ways to determine its Euler characteristic, its connectivity, its orientability class and the number of its boundary components. Hence, its genus (if orientable, the number of handles; and if non-orientable, the number of cross caps) also can be determined.

3. AVERAGE LENGTH ESTIMATES

In this section we give a tool which is very useful in working with decision problems and algorithms, especially those related to Dehn fillings. We call it an average length estimate; it is used extensively in [11]. If M is a 3-manifold with triangulation \mathcal{T} and F is a properly embedded surface in M and is in general position with respect to $\mathcal{T}^{(2)}$, then we have $L(\partial F)$ defined. If b is the number of components of ∂F , then we set $\lambda_{av} = L(\partial F)/b$ and say λ_{av} is the *average length (of the components) of ∂F* . Under various conditions placed on the topology of a manifold M , the average length estimate says that for any triangulation \mathcal{T} of M , there is a constant C , depending only on M and \mathcal{T} , so that all properly embedded, essential surfaces of bounded genus have the average length of their boundary bounded by C . Hence, depending only on the manifold and a given triangulation, there is a number so that all essential surfaces of bounded genus must have a short boundary component. We give several variants here and add another useful variant within the proof of Theorem 5.9.

If a 3-manifold has an essential annulus, then often it is possible by Dehn twisting about such an annulus to obtain surfaces with all boundary components being arbitrarily long; the surfaces are homeomorphic and so all have a fixed genus. Hence, there are no preassigned values for short boundaries. There also are examples of families of surfaces of fixed genus in link-manifolds, where for any value there is a surface in the family having some boundary component of length larger than this value; this can happen in a link-manifold with or without having an essential annulus between distinct boundary components. However, if there are no such essential annuli, then each such surface must also have short boundary components. We say the 3-manifold M is *anannular* if there are no properly embedded, essential annuli in M .

3.1. Proposition. *Suppose M is a compact, irreducible, ∂ -irreducible and anannular 3-manifold with triangulation \mathcal{T} . There is a constant $C = C(M, \mathcal{T})$, depending only on M and \mathcal{T} , so that if F is an embedded, essential, normal surface that is least weight in its isotopy class and λ_{av} is the average length of the components of ∂F , then $\lambda_{av} \leq C(2g + 1)$, where g is the genus of F .*

Proof. Since F is essential and least weight in its isotopy class, we have from [6] and [12] that any normal surface that projects into $\mathcal{C}(F)$ is essential. In particular, from the hypotheses on M , no normal surface in $\mathcal{C}(F)$ is a 2-sphere, projective plane, annulus or Möbius band. Hence, the normal surface F can be written as a sum $F = \sum n_i F_i + \sum m_j K_j$, where each F_i is fundamental and essential and each K_j is an essential torus or Klein bottle.

Set

$$C = \frac{L(\partial F_i)}{-\chi(F_i)},$$

where F_i is a fundamental surface for (M, \mathcal{T}) and $\chi(F_i) < 0$.

Then we have

$$b\lambda_{av} = L(\partial F) = \sum n_i L(\partial F_i) \leq C \sum n_i (-\chi(F_i)) = C(-\chi(F)) = C(2g - 2 + b),$$

where b is the number of components of ∂F . It follows that $\lambda_{av} \leq C(2g + 1)$. \square

We have noted that the possibility of Dehn twisting a surface F about a properly embedded annulus in the manifold can lengthen the boundary of F but does not

change its genus. However, after passing to normal surfaces, this feature can only happen if such an annulus is fundamental and also has the same quadrilateral type as the surface F . Also, Dehn twisting a surface F about an annulus having its boundary in a single boundary component that is a torus does not change the slope of ∂F . We have two variants to Proposition 3.1 given as the next proposition and its corollary. In addition, we point out that while Euler characteristic is additive under geometric sum, genus, in general, is not. The problem, of course, is that if a normal surface is a geometric sum of other normal surfaces, then it is not always true that the number of boundaries of the sum is the sum of the number of boundaries of the summands. But in many cases Theorem 2.3 does allow this to happen and thus gives us results when the manifold has tori boundary that we would not get otherwise. This is quite evident in the next two results.

3.2. Proposition. *Suppose M is a link-manifold with no embedded annuli having essential boundary curves in distinct components of ∂M . Furthermore, suppose \mathcal{T} is a 0-efficient triangulation of M . Then there is a constant $C = C(M, \mathcal{T})$, depending only on M and \mathcal{T} , so that if F is an embedded normal surface in M with no trivial boundary curves and λ_{av} is the average length of the components of ∂F , then $\lambda_{av} \leq 2C(g + 1)$, where g is the genus of F .*

Proof. If F is an embedded normal surface with no trivial boundary curves, then we can write

$$F = \sum l_i F_i + \sum m_j K_j + \sum n_k A_k,$$

where each summand is fundamental, $\chi(F_i) < 0$, K_j is either a torus or Klein bottle and A_k is either a Möbius band or an annulus with both its boundary curves in the same component of ∂M . We let $|A_k|$ denote the number of boundary components of A_k and set

$$C = \max \left\{ \frac{L(\partial F_1)}{-\chi(F_1)}, \dots, \frac{L(\partial F_I)}{-\chi(F_I)}, \frac{L(\partial A_1)}{|A_1|}, \dots, \frac{L(\partial A_K)}{|A_K|} \right\}.$$

Let $F' = \sum l_i F_i$; then $\chi(F) = \chi(F')$.

Now, by Theorem 2.3, we have that if an annulus or Möbius band summand and F' meet the same boundary torus of M , then their boundaries have the same slope in this boundary torus. Hence, if $b_{F'}$ is the number of boundary components of $\partial F'$, $b_A = \sum n_k |A_k|$, is the number of boundary components of $\sum n_k A_k$ and b is the number of boundary components of F , then $b = b_{F'} + b_A$. It follows that

$$\begin{aligned} b\lambda_{av} &= L(\partial F) = \sum l_i L(\partial F_i) + \sum n_k L(\partial A_k) \leq \\ &\leq C \left(\sum l_i (-\chi(F_i)) + \sum n_k |\partial A_k| \right) = C(-\chi(F') + b_A). \end{aligned}$$

However, $\chi(F) = \chi(F')$; hence, $b\lambda_{av} \leq C(-\chi(F) + b_A) = C(2g - 2 + b) + Cb_A$. Therefore,

$$\lambda_{av} \leq C \left(\frac{(2g - 2)}{b} + 1 \right) + C \left(\frac{b_A}{b} \right) \leq C(2g + 2).$$

□

Suppose M is a 3-manifold and \mathcal{T} is a triangulation of M . We shall say a finite collection of embedded normal surfaces $\{H_1, \dots, H_m\}$ is a *spanning collection* for the embedded normal surfaces in (M, \mathcal{T}) if and only if for any embedded normal

surface F in M , we have $F = \sum n_i H_i$, where n_i is a nonnegative integer. The fundamental surfaces are a spanning collection.

The following lemma is a corollary of the proof of the Proposition 3.2. In this corollary, we substitute a spanning collection for the fundamental surfaces; hence, the constant C becomes dependent on the spanning collection (a fundamental collection is unique; whereas, there are possibly many choices of distinct spanning collections).

3.3. Corollary. *Suppose M is link-manifold, \mathcal{T} is a minimal-vertex triangulation of M and \mathcal{H} is a spanning collection of normal surfaces. There is a constant $C = C(M, \mathcal{T}, \mathcal{H})$, depending only on M , \mathcal{T} and \mathcal{H} , so that if F is an embedded normal surface in M with no trivial boundary curves and F can be written as a geometric sum $F = \sum l_i H_i$, where $H_i \in \mathcal{H}$ and either $\chi(H_i) < 0$ or H_i is a torus or Klein bottle, then*

$$\lambda_{av} \leq C(2g + 1),$$

where λ_{av} is the average length of the components of ∂F and g is the genus of F .

4. FINDING PLANAR SURFACES

In this section we give an algorithm to decide if a given link-manifold contains a properly embedded, essential, planar surface. An important aspect of this algorithm is, unlike those in Section 2, we do not show that if there is such a planar surface, then there is one among the fundamental surfaces of the given triangulation. Indeed, we must construct a family that, while still finite, goes beyond the fundamental surfaces of the given triangulation. The background material and other algorithms we use in the proof of this result typically assume the given manifold is known to be irreducible and ∂ -irreducible. So, we begin with an algorithm that transforms the problem for the given manifold to a possibly distinct but constructible manifold that is known to be irreducible and ∂ -irreducible.

Beginning in the most general situation, we are given a 3-manifold via a triangulation. We can easily check that it is a link-manifold; however, it very well may be ∂ -reducible or reducible. Recall that if a link-manifold M is ∂ -reducible and is not a solid torus, then it also is reducible and can be written as a connected sum, $M = (D^2 \times S^1) \# M'$, of a solid torus and a 3-manifold M' . There is an algorithm, due to W. Haken [3], which in Lemma 2.6 we adapted to the generality we are using here, that will decide if a given 3-manifold is ∂ -reducible. If it is, the algorithm will construct an essential disk. Hence, we could begin by running this algorithm. Having run this algorithm, if the given link-manifold is ∂ -reducible, we have found a properly embedded, essential planar surface and we are done. However, if the manifold is not ∂ -reducible, it may still be reducible. Thus it would still be necessary to undertake the construction of a prime decomposition. So, we do this first, including, with little extra work, an algorithm that will modify a given triangulation of a manifold that is known to be irreducible and ∂ -irreducible to a triangulation of the manifold that is 0-efficient. We note, however, there is a version of the Haken algorithm within these algorithms. Also, we do not really need that triangulations are 0-efficient; but rather use very strongly that the given manifolds are irreducible and ∂ -irreducible and are given by minimal-vertex triangulations.

If Q is a compact 3-manifold, we use the notation $p(Q)$ to denote $Q \# \cdots \# Q$, where there are $p \geq 0$ copies of Q .

4.1. Theorem. [9] *Given a link-manifold M via a triangulation \mathcal{T} , there is an algorithm to construct a prime decomposition*

$$M = p(S^2 \times S^1) \# q(\mathbb{R}P^3) \# r(D^2 \times S^1) \# M_1 \# \cdots \# M_n,$$

where p, q and r are nonnegative integers and each M_i is given by a 0-efficient triangulation $\mathcal{T}_i, i = 1, \dots, n$, respectively; furthermore, $\sum_{i=1}^n \text{Card}(\mathcal{T}_i) \leq \text{Card}(\mathcal{T})$.

Proof. First, from [7] and Proposition 5.7 of [9], there is an algorithm to decide if there are non vertex-linking normal 2-spheres. If there is one, the algorithm constructs one. We then proceed as in the proof of Theorem 5.9 of [9] to crush the triangulation along a suitable non vertex-linking normal 2-sphere. In the case of a link-manifold, if there is a non vertex-linking normal 2-sphere, there is one along which we can crush. The process repeats as long as there is a non vertex-linking normal 2-sphere. After each crushing, we reduce the number of tetrahedra we had before crushing, thus the process must stop in a finite number of steps at which point the algorithm has managed to find any factors that are $S^2 \times S^1$ or $\mathbb{R}P^3$ and we have a connected sum decomposition

$$M = p(S^2 \times S^1) \# q(\mathbb{R}P^3) \# M'_1 \# \cdots \# M'_n,$$

where each factor M'_i is given by a triangulation in which the only normal 2-spheres are vertex-linking. We note that the triangulations of the closed factors in this decomposition are 0-efficient; and by using the 3-sphere recognition algorithm [15, 17] (also, see [9], Theorem 5.11), we may assume no factor is S^3 .

If the only normal 2-spheres are vertex-linking, then we consider the factors M'_i in the above connected sum decomposition that have nonempty boundary. For any such factor, we determine if there are any non vertex-linking normal disks. Again, from [7] and [9] there is an algorithm to decide if there are any non vertex-linking normal disks. If there is one, then the algorithm constructs one. However, here there is a slight twist to the argument in [9]; namely, we may have a non separating essential disk. Since each factor is irreducible, we conclude that a factor with a non separating disk is a solid torus and contributes a factor in the prime decomposition of the form $D^2 \times S^1$. If the disk is separating, we proceed as in the proof of Theorem 5.17 of [9] to crush the triangulation along a suitable separating, non vertex-linking normal disk. The process repeats as long as there is a non vertex-linking normal disk. Again, since at each crushing we must reduce the number of tetrahedra we had before crushing, the process must stop in a finite number of steps.

Upon having no non vertex-linking normal disk, we have the desired decomposition, where the factors M_1, \dots, M_n are given by 0-efficient triangulations $\mathcal{T}_1, \dots, \mathcal{T}_n$, respectively. Furthermore, the total number of tetrahedra in the triangulations $\mathcal{T}_1, \dots, \mathcal{T}_n$ is no larger than the number we started with in \mathcal{T} and is equal if and only if \mathcal{T} is itself 0-efficient. \square

Knowing that we can construct a prime decomposition of a given manifold, we make the following observation, a proof of which is easily derived from classical 3-manifold “cut-and-paste” methods.

4.2. Proposition. *The 3-manifold M contains a properly embedded, essential, planar surface if and only if one of the prime factors of M contains a properly embedded, essential, planar surface. Moreover, if B is a component of ∂M , then M contains an embedded punctured disk with boundary in B and punctures in $\partial M \setminus B$ if*

and only if the prime factor, say M_B of M containing B has an embedded punctured-disk with boundary in B and punctures in $\partial M_B \setminus B$.

If M is a link-manifold, then the decomposition given in Theorem 4.1 necessarily has prime factors that are known to be solid tori or are given by 0-efficient triangulations and thus are known to be irreducible and ∂ -irreducible link-manifolds. We ignore any factors that are closed.

We now have a special case of the main theorem of this section; which also is needed in the proof of the main theorem. Notice that in Lemma 4.3, we do not require that the link-manifold M be irreducible or ∂ -irreducible and we do not require that we consider only essential, planar surfaces. We do, however, require that all of the boundary components of our planar surfaces are essential curves in a single component B of ∂M . There is, of course, something subtle here, as it is quite easy to find an embedded, planar, normal surface with all of its boundary components essential curves in a component B of ∂M . For example, consider the surface one obtains by taking the frontier of a small regular neighborhood of an edge in the component B of ∂M . Often, this surface is normal and fundamental. If it is not normal, it can be shrunk, using a barrier surface argument [9], it either normalizes to an embedded, normal, planar surface (annulus) with its boundary essential curves in B or it follows that M is a solid torus. The point, which will give us a nontrivial conclusion in Lemma 4.3, is that the existence of a certain special planar, normal surface, say P , guarantees not only the existence of a similar planar, fundamental normal surface but one that also projects into $\mathcal{C}(P)$, the carrier of P . Following the statement and proof in this special case, we give two corollaries. The second corollary is our main theorem of this section for knot-manifolds.

4.3. Lemma. *Suppose M is a link-manifold with a minimal-vertex triangulation \mathcal{T} and B is a component of ∂M . If there is an embedded, planar, normal surface P with all its boundary essential curves in B and if P can be written as a sum of fundamental surfaces, none of which is a 2-sphere or projective plane, then there is an embedded, planar, normal surface with all its boundary essential curves in B that projects into a fundamental class in $\mathcal{C}(P)$, the carrier of P .*

Proof. Suppose there is an embedded, planar, normal surface P with all of its boundary essential curves in B and

$$P = \sum n_i F_i + \sum n'_j F'_j,$$

where F_i is fundamental, orientable and not a 2-sphere and F'_j is fundamental, non-orientable and not a projective plane. Considering Euler characteristics, we have for b the number of boundary components of P , b_i, b'_j the number of boundary components of F_i, F'_j , respectively,

$$\begin{aligned} b - 2 &= -\chi(P) = \sum n_i (-\chi(F_i)) + \sum n'_j (-\chi(F'_j)) = \\ &= 2 \left(\sum n_i (g_i - 1) + \sum n'_j \left(\frac{c_j}{2} - 1 \right) \right) + \sum n_i b_i + \sum n'_j b'_j, \end{aligned}$$

where g_i is the genus of F_i and c_j is the number of cross caps in F'_j . Since all the boundary curves of P are in B , we have all the boundary curves of the surfaces F_i and F'_j also in B . We have all the surfaces F_i, F'_j and P satisfying the same quadrilateral conditions; hence, by Theorem 2.3 the slopes of the boundary curves are complementary. But since the components of ∂P are essential curves in B , all

the surfaces actually have the same boundary slope and this slope is the slope of the boundary of P . It follows that $b = \sum n_i b_i + \sum n'_j b'_j$ and therefore

$$-1 = \sum n_i (g_i - 1) + \sum n'_j \left(\frac{c_j}{2} - 1 \right).$$

We conclude some $g_i = 0$ or some $c_j = 1$. If $g_i = 0$, then F_i is a fundamental planar surface (no F_i is a 2-sphere); if $c_j = 1$, then F'_j is a fundamental Möbius band (no F'_j is a projective plane). All summands in the geometric sum of P must project into $\mathcal{C}(P)$. \square

4.4. Corollary. *Given a link-manifold M , there is an algorithm to decide if M contains a properly embedded, essential, planar surface with all its boundary in a given component B of ∂M . If there is one, the algorithm will construct one.*

Proof. We are given the link-manifold M via a triangulation \mathcal{T} . By Theorem 4.1, we construct a prime decomposition

$$P = p(S^2 \times S^1) \# q(\mathbb{R}P^3) \# r(D^2 \times S^1) \# M_1 \# \cdots \# M_n,$$

where each M_i has a 0-efficient triangulation. If $r \neq 0$ and B is a component of a solid torus factor, $D^2 \times I$, then we have an essential disk with boundary in B and the algorithm that constructs the prime decomposition will construct such an essential disk. So, we shall assume B is a component of some M_i , say M_1 .

By Lemma 4.2, M contains a properly embedded, essential, planar surface with all its boundary in B if and only if M_1 contains a properly embedded, essential, planar surface with all its boundary in B . Furthermore, if M_1 contains such a surface then it contains a normal one; and from Theorem 2.9, if P is the least weight properly embedded, essential, planar normal surface with all its boundary in B , then every surface in $\mathcal{C}(P)$ is essential. Since M_1 has a 0-efficient triangulation, P can be written as a sum of fundamental normal surfaces, each must project into $\mathcal{C}(P)$, and none can be a 2-sphere or a projective plane. Thus by Lemma 4.3, there is a planar surface projectively equivalent to the projection of a fundamental surface into $\mathcal{C}(P)$. Every surface that projects into $\mathcal{C}(P)$ is essential. \square

4.5. Corollary. *Given a knot-manifold there is an algorithm to decide if it contains a properly embedded, essential, planar surface; if it does, the algorithm will construct one.*

Remark. If in either of the situations given in Corollaries 4.4 or 4.5 we know the given manifold is irreducible and if it is given by a minimal-vertex triangulation \mathcal{T} , then it contains a properly embedded, essential, planar surface if and only if there is an embedded, essential planar, normal surface that is fundamental in (X, \mathcal{T}) .

4.6. Theorem. *Given a link-manifold there is an algorithm to decide if it contains a properly embedded, essential, planar surface; if it does, the algorithm will construct one.*

Proof. From the above, we may assume we are given a link-manifold M via a 0-efficient triangulation \mathcal{T} .

If there is a properly embedded, essential, planar surface in M , then for any triangulation of M there is a normal such surface and, therefore, an embedded, essential, planar, normal surface that is least weight in its isotopy class.

The plan of the proof is to study what the situation is under the assumption that there is such a normal surface, say P , in M . We have from Corollary 4.5, that

for this situation in a knot-manifold, there is an essential, planar surface among the fundamental surfaces for (X, \mathcal{T}) . This does not seem to be necessarily true for link-manifolds. So, we start by constructing the fundamental surfaces of M . If we find an essential, planar normal surface among these surfaces, then we are done; if we do not find an essential, planar surface among the fundamental surfaces of M , we work our way via Dehn fillings toward a knot-manifold where we then, hopefully, can solve the problem. The first issue is to determine those Dehn fillings we should use. For this we use the method of average length estimates, which depend only on the triangulation when considering fixed genus (planar) surfaces. However, we immediately run into the classical issues with Dehn fillings; namely, after Dehn filling we may lose irreducibility, ∂ -irreducibility and that the surfaces we are interested in are essential. To handle these problems, we use triangulated Dehn fillings. This avoids ever having to re-triangulate M and enables us to understand the normal surfaces in the Dehn filled manifold relative to normal surfaces in the manifold before Dehn filling. In this way, if the link-manifold has n tori in its boundary, then in a succession of no more than n steps, we construct at most n finite collections of normal surfaces in (X, \mathcal{T}) , showing that there is a properly embedded, essential, planar surface in M if and only if there is an essential, planar, normal surface in the collection of surfaces we construct.

The notation is a bit tricky. At each new step, we construct a finite family of triangulated Dehn fillings for each member of a previously constructed finite family of triangulated Dehn fillings. Then for each new construction, we compute the fundamental normal surfaces in the new triangulated Dehn filling and select from these a subcollection, the members of which have a particularly nice decomposition in terms of that Dehn filling. These decompositions provide normal surfaces in M that become part of our desired collection of normal surfaces in (M, \mathcal{T}) . The triangulation \mathcal{T} of M in all these triangulated Dehn fillings remains constant.

We have provide Figure 4 as an example having at most three steps.

To this end, we suppose P is an embedded, essential, planar, normal surface and is least weight in its isotopy class. Let $\mathcal{C}(P)$ denote the carrier of P . Then by Theorem 2.9 (see [6] and [12]) any normal surface in (M, \mathcal{T}) that is projectively equivalent to a surface in $\mathcal{C}(P)$ is essential in M .

Consider the fundamental surfaces in (M, \mathcal{T}) with projective class in $\mathcal{C}(P)$. Such a surface is essential in M and can not be a disk, $\mathbb{R}P^2$ or S^2 . If any is planar, then there is an embedded, essential planar surface among the fundamental surfaces of (M, \mathcal{T}) . If either M is a knot-manifold or P has all its boundary in a single component of ∂M , then by the Remark following Proposition 4.3 and its corollaries, we would necessarily discover an embedded planar, normal surface among the fundamental surfaces of (M, \mathcal{T}) .

Having made these observations, we consider the first step of a general algorithm. Given a link-manifold M with 0-efficient triangulation \mathcal{T} , we first construct the fundamental surfaces of (M, \mathcal{T}) . This gives us a constructible, finite collection, \mathcal{F}^0 , of normal surfaces in M . We can recognize any of these surfaces that are planar; and by Theorem 2.8, we can decide if any of these are essential in M . If P were to exist and if any of the fundamental surfaces of (M, \mathcal{T}) with projective class in $\mathcal{C}(P)$ were planar, then we would have found one among the surfaces in the collection \mathcal{F}^0 .

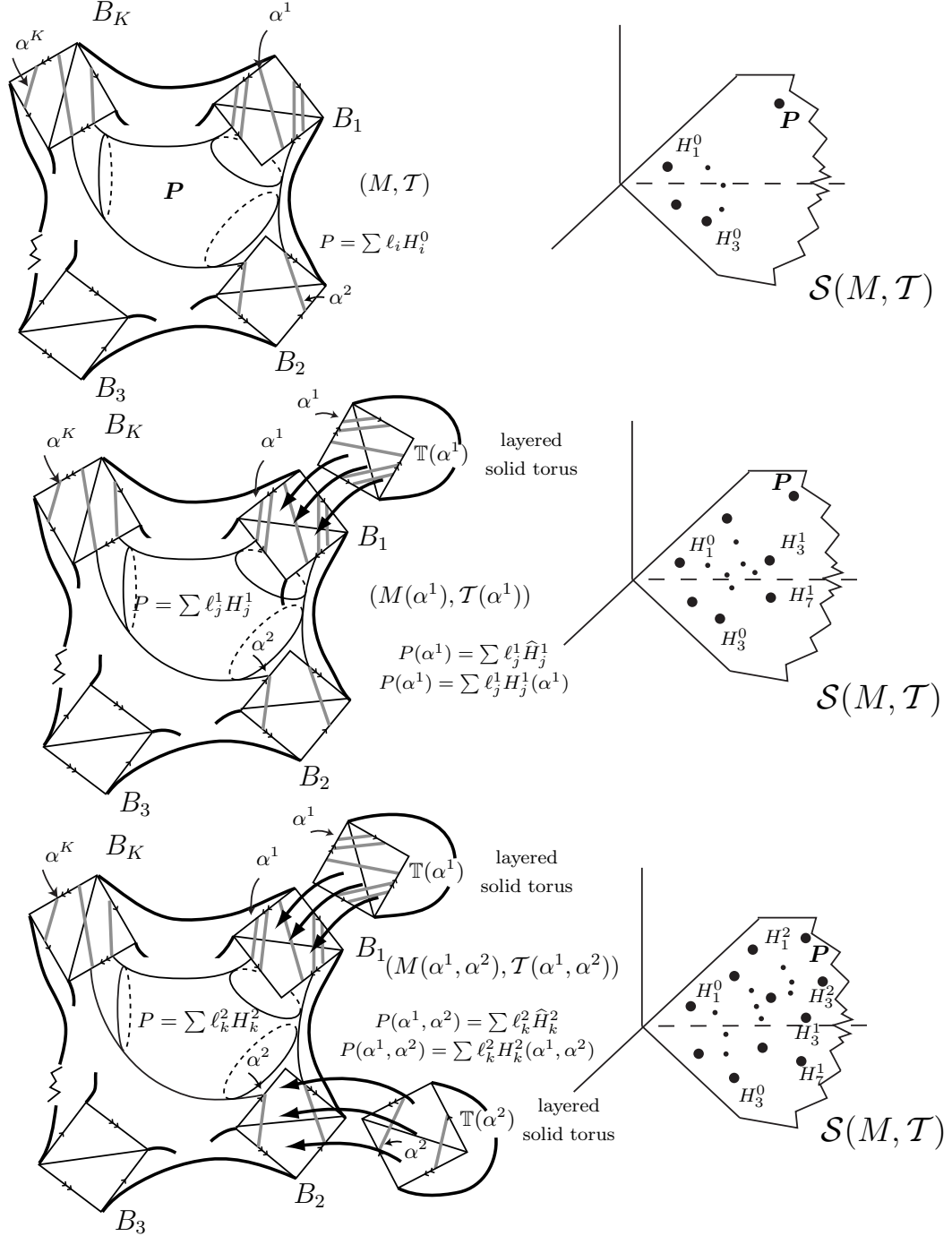


FIGURE 4. First the planar surface P is written as a sum of fundamentals H_i^0 in (M, T) ; then we re-write P as a sum $P = \sum \ell_j^1 H_j^1$, where the H_j^1 are determined by fundamentals in $(M(\alpha^1), T(\alpha^1))$; and finally we re-write P as a sum $P = \sum \ell_k^2 H_k^2$, where the H_k^2 are determined by fundamentals in $(M(\alpha^1, \alpha^2), T(\alpha^1, \alpha^2))$. There is an essential planar surface in M iff there is one in one of the finite collections $\{H_i^0\}, \{H_j^1\}$ or $\{H_k^2\}$.

Now, we return to our consideration should P exist. From the preceding paragraphs, we may assume none of the fundamental surfaces of (M, \mathcal{T}) with projective class in $\mathcal{C}(P)$ are planar surfaces. Then it follows that P can be written as a sum

$$P = \sum l_i H_i^0,$$

where H_i^0 is fundamental, $\chi(H_i^0) \leq 0$ and if $\chi(H_i^0) = 0$, then H_i^0 is a torus or Klein bottle. Set

$$C'_0 = \max \left\{ \frac{L(\partial H_i^0)}{-\chi(H_i^0)} \right\},$$

where $\chi(H_i^0) < 0$ and is fundamental in (M, \mathcal{T}) with projective class in $\mathcal{C}(P)$. We have from Corollary 3.3 that $\lambda_{av}^0 < C'_0$, where λ_{av}^0 is the average length of the components of ∂P .

Hence, there is a component of ∂P , having slope (say) α^1 , with the property that $L(\alpha^1) \leq C'_0$, where $L(\alpha^1)$ is the length of α^1 (there is a unique norml representative of the slope α^1). All components of ∂P in the same component of ∂M as α^1 also have slope α^1 . Let $M(\alpha^1)$ denote the triangulated Dehn filling of M along slope α^1 ; $M(\alpha^1) = M \cup \mathbb{T}(\alpha^1)$ has a triangulation $\mathcal{T}(\alpha^1)$ that is \mathcal{T} on M and a minimal layered-triangulation on the solid torus $\mathbb{T}(\alpha^1)$. From P we get a planar normal surface, denoted $P(\alpha^1)$, in $(M(\alpha^1), \mathcal{T}(\alpha^1))$ obtained by “capping off” the surface P with copies of the (normal) meridional disk in the solid torus $\mathbb{T}(\alpha^1)$.

We can write $P(\alpha^1) = \sum \ell_j^1 \hat{H}_j^1$, where the surfaces \hat{H}_j^1 are fundamental in $(M(\alpha^1), \mathcal{T}(\alpha^1))$. But since we are using triangulated Dehn fillings, it follows from Lemma 2.5 that each $\hat{H}_j^1 = H_j^1(\alpha^1)$, where H_j^1 is a normal surface in (M, \mathcal{T}) and $H_j^1(\alpha^1)$ is H_j^1 capped off with copies of the meridional disk in $\mathbb{T}(\alpha^1)$. Thus $P(\alpha^1) = \sum \ell_j^1 H_j^1(\alpha^1)$ and we can *re-write* P as $P = \sum \ell_j^1 H_j^1$, where H_j^1 may not be fundamental in M but $H_j^1(\alpha^1)$ is fundamental in $M(\alpha^1)$. Note that each H_j^1 meets the component of ∂M containing α^1 in the slope α^1 and H_j^1 projects into $\mathcal{C}(P)$.

For $P(\alpha^1) = \sum \ell_j^1 H_j^1(\alpha^1)$, if any $H_j^1(\alpha^1)$ is $S^2, \mathbb{R}P^3$ or planar, then H_j^1 is planar (there are no normal 2-spheres or projective planes in M) and since H_j^1 projects into $\mathcal{C}(P)$ it would be essential. In this case, we would have an essential, planar surface in M among the surfaces $\{H_j^1\}$.

Now, we consider the second step of a general algorithm. Having \mathcal{F}^0 , the fundamental surfaces in (M, \mathcal{T}) , if there are no essential, planar, fundamental surfaces, we compute

$$C_0 = \max \left\{ \frac{L(\partial G_i^0)}{-\chi(G_i^0)} \right\},$$

where $\chi(G_i^0) < 0$ and G_i^0 is fundamental in (M, \mathcal{T}) , $G_i^0 \in \mathcal{F}^0$. Next we find all slopes $\alpha_1^1, \dots, \alpha_{N_1}^1$ on ∂M with $L(\alpha_p^1) \leq C_0$, where $L(\alpha_p^1)$ is the length of a normal representative of the slope α_p^1 .

We construct the triangulated Dehn fillings $(M(\alpha_p^1), \mathcal{T}(\alpha_p^1))$ for all slopes $\alpha_1^1, \dots, \alpha_{N_1}^1$; then in each of these we compute the fundamental surfaces. From this entire collection of fundamental surfaces, we select those in the various $(M(\alpha_p^1), \mathcal{T}(\alpha_p^1))$ that can be written $G_j^1(\alpha_p^1)$ for G_j^1 a normal surface in (M, \mathcal{T}) ; namely, we define \mathcal{F}^1 so that $G_j^1 \in \mathcal{F}^1$ if and only if there is a fundamental surface \hat{G}_j^1 in $(M(\alpha_p^1), \mathcal{T}(\alpha_p^1))$, for some α_p^1 , and $\hat{G}_j^1 = G_j^1(\alpha_p^1)$, where G_j^1 is normal in M . The fundamental surface

\widehat{G}_j^1 meets the solid torus $\mathbb{T}(\alpha_p^1)$ only in meridional disks and so is some G_j^1 in M “capped off”; in particular, G_j^1 has a boundary component with slope α_p^1 .

We note that $C'_0 \leq C_0$; and therefore, the slope α^1 above is a slope α_p^1 for some p and the surfaces H_p^1 considered above are among the surfaces G_j^1 .

This gives us a second constructible, finite collection, \mathcal{F}^1 , of normal surfaces in M . We can recognize any of these surfaces that are planar and essential in M . Again, if P were to exist and if any of the surfaces in \mathcal{F}^1 with projective class in $\mathcal{C}(P)$ were planar, then we would find an essential planar surface in the collection \mathcal{F}^1 . Note that if either M has only two boundary components or $P(\alpha^1)$ has all its boundary in a single component of $\partial M(\alpha^1)$, then by the Remark following Proposition 4.3 and its corollaries, we would necessarily discover an embedded, essential, planar normal surface in \mathcal{F}^1 .

Now, suppose M has n boundary components (at this point we may suppose $n > 2$) and m is an integer, $1 \leq m < n$. We assume we have determined m collections of slopes on ∂M , $\{\alpha_1^1, \dots, \alpha_{N_1}^1\}, \dots, \{\alpha_1^m, \dots, \alpha_{N_m}^m\}$, along with $m+1$ collections of normal surfaces in (M, \mathcal{T}) , $\mathcal{F}^0, \dots, \mathcal{F}^m$, so that \mathcal{F}^k is the collection of normal surfaces of (M, \mathcal{T}) and for each k , $1 \leq k \leq m$, the normal surface $G_i^k \in \mathcal{F}^k$ if and only if there is a fundamental surface \widehat{G}_i^k in $(M(\alpha_{j_1}^1, \dots, \alpha_{j_k}^k), \mathcal{T}(\alpha_{j_1}^1, \dots, \alpha_{j_k}^k))$ for some set of slopes $\alpha_{j_1}^1, \dots, \alpha_{j_k}^k$ with $\alpha_{j_i}^i \in \{\alpha_1^i, \dots, \alpha_{N_i}^i\}$ and $\widehat{G}_i^k = G_i^k(\alpha_{j_1}^1, \dots, \alpha_{j_k}^k)$. Furthermore, if there is an embedded, essential, planar, normal surface in M , then either there is an embedded, essential, planar, normal surface in one of the collections \mathcal{F}^k or there is an embedded, essential, planar, normal surface P in M that is least weight in its isotopy class and for some set of slopes $\{\alpha^1, \dots, \alpha^m\}$, we can cap off P along these slopes to get the planar surface $P(\alpha^1, \dots, \alpha^m)$, where $\alpha^i \in \{\alpha_1^i, \dots, \alpha_{N_i}^i\}$, $1 \leq i \leq m$.

This gives us a finite collection, $\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^m$ of normal surfaces in M . We can recognize among these surfaces any that are planar and essential in M . Again, we observe that if P were to exist and if any of the surfaces in \mathcal{F}^i , $0 \leq i \leq m$, with projective class in $\mathcal{C}(P)$ were planar, then we would find an essential planar surface in the collection $\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^m$. Note that if either $n = m+1$ ($M(\alpha^1, \dots, \alpha^m)$ has only one boundary component) or $P(\alpha^1, \dots, \alpha^m)$ has all its boundary in a single component of $\partial M(\alpha^1, \dots, \alpha^m)$, then we would necessarily discover an embedded planar, normal surface in $\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^m$.

We now return to what the situation might be should P exist and we have not found an embedded, essential, planar surface in the collection $\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^m$. Consider $P(\alpha^1, \dots, \alpha^m)$. We can write $P(\alpha^1, \dots, \alpha^m) = \sum \ell_k^m \widehat{H}_k^m$, where the surfaces \widehat{H}_k^m are fundamental in $(M(\alpha^1, \dots, \alpha^m), \mathcal{T}(\alpha^1, \dots, \alpha^m))$. Again, since each \widehat{H}_k^m meets the solid tori $\mathbb{T}(\alpha^i)$ in surfaces that sum to the meridional disks, it follows that each $\widehat{H}_k^m = H_k^m(\alpha^1, \dots, \alpha^m)$, where H_k^m is a normal surface in (M, \mathcal{T}) and $H_k^m(\alpha^1, \dots, \alpha^m)$ is H_k^m capped off with copies of the meridional disks in the solid tori $\mathbb{T}(\alpha^1), \dots, \mathbb{T}(\alpha^m)$. Thus $P(\alpha^1, \dots, \alpha^m) = \sum \ell_k^m H_k^m(\alpha^1, \dots, \alpha^m)$ and we can re-write P as $P = \sum \ell_k^m H_k^m$, where H_k^m may not be fundamental in M but $H_k^m(\alpha^1, \dots, \alpha^m)$ is fundamental in $M(\alpha^1, \dots, \alpha^m)$. Note that each H_k^m meets the component of ∂M containing the slope α^i in that slope and H_k^m projects into $\mathcal{C}(P)$. In particular, the collection $\{H_k^m\}$ is a spanning collection for the normal surfaces in M over $\mathcal{C}(P)$, the carrier of P .

If any $H_k^m(\alpha^1, \dots, \alpha^m)$ is $S^2, \mathbb{R}P^3$ or planar, then H_k^m is planar (there are no normal 2-spheres or projective planes in M) and since H_k^m projects into $\mathcal{C}(P)$ it would be essential. In this case, we would have an essential, planar surface in M among the surfaces $\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^m$.

It follows, from our assumption that there are no essential planar surfaces in the collections $\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^m$, that P can be written as a sum

$$P = \sum l_k H_k^m,$$

where $\chi(H_k^m) \leq 0$ and if $\chi(H_k^m) = 0$, then H_k^m is a torus or Klein bottle.

Set

$$C'_m = \max \left\{ \frac{L(\partial H_k^m(\alpha^1, \dots, \alpha^m))}{-\chi(H_k^m(\alpha^1, \dots, \alpha^m))} \right\},$$

where $\chi(H_k^m(\alpha^1, \dots, \alpha^m)) < 0$ and is fundamental with projective class in $\mathcal{C}(P(\alpha^1, \dots, \alpha^m))$. We have from Corollary 3.3 that $\lambda_{av}^m \leq C'_m$, where λ_{av}^m is the average length of the components of $\partial P(\alpha^1, \dots, \alpha^m)$.

Hence, there is a component of $\partial P(\alpha^1, \dots, \alpha^m)$ having slope (say) α^{m+1} with the property that $L(\alpha^{m+1}) \leq C'_m$, where $L(\alpha^{m+1})$ is the length of α^{m+1} . All components of $\partial P(\alpha^1, \dots, \alpha^m)$ in the same component of $\partial M(\alpha^1, \dots, \alpha^m)$ as α^{m+1} also have slope α^{m+1} . Let $M(\alpha^1, \dots, \alpha^m, \alpha^{m+1})$ denote the triangulated Dehn filling of $M(\alpha^1, \dots, \alpha^m)$ along slope α^{m+1} ; $M(\alpha^1, \dots, \alpha^m, \alpha^{m+1}) = M(\alpha^1, \dots, \alpha^m) \cup \mathbb{T}(\alpha^{m+1})$ has a triangulation $\mathcal{T}(\alpha^1, \dots, \alpha^m, \alpha^{m+1})$ that is \mathcal{T} on M and a minimal layered-triangulation on each $\mathbb{T}(\alpha^i)$, $i = 1, \dots, m+1$. From P we get a planar normal surface, denoted $P(\alpha^1, \dots, \alpha^m, \alpha^{m+1})$, in $(M(\alpha^1, \dots, \alpha^m, \alpha^{m+1}), \mathcal{T}(\alpha^1, \dots, \alpha^m, \alpha^{m+1}))$ obtained by “capping off” the surface P with copies of the (normal) meridional disks in the solid tori $\mathbb{T}(\alpha^i)$.

We can write $P(\alpha^1, \dots, \alpha^m, \alpha^{m+1}) = \sum \ell_r^{m+1} \hat{H}_r^{m+1}$, where the surfaces \hat{H}_r^{m+1} are fundamental in $(M(\alpha^1, \dots, \alpha^m, \alpha^{m+1}), \mathcal{T}(\alpha^1, \dots, \alpha^m, \alpha^{m+1}))$. But just as above, each $\hat{H}_r^{m+1} = H_r^{m+1}(\alpha^1, \dots, \alpha^m, \alpha^{m+1})$. Thus we re-write P as $P = \sum \ell_r^{m+1} H_r^{m+1}$, where H_r^{m+1} may not be fundamental in M but $H_r^{m+1}(\alpha^1, \dots, \alpha^m, \alpha^{m+1})$ is fundamental in $M(\alpha^1, \dots, \alpha^m, \alpha^{m+1})$. Note that each H_r^{m+1} meets the component of ∂M containing the slope α^{m+1} in the slope α^{m+1} and H_r^{m+1} projects into $\mathcal{C}(P)$.

If any $H_r^{m+1}(\alpha^1, \dots, \alpha^m, \alpha^{m+1})$ is $S^2, \mathbb{R}P^3$ or planar, then H_r^{m+1} is planar and as earlier, we would have an essential, planar surface in M among the surfaces H_r^{m+1} .

Finally, this brings us to the completion of the induction step, going from m to $m+1$ in our construction of the family \mathcal{F}^{m+1} , having the families $\mathcal{F}^0, \dots, \mathcal{F}^m$.

Compute

$$C_m = \max \left\{ \frac{L(\partial G_r^m(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m))}{-\chi(G_r^m(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m))} \right\},$$

where $\chi(G_r^m(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m)) < 0$ and $G_r^m(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m)$ is fundamental in $(M(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m), \mathcal{T}(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m))$ for some set of slopes $\alpha_{j_1}^1, \dots, \alpha_{j_m}^m$ with $\alpha_{j_i}^i \in \{\alpha_1^i, \dots, \alpha_{N_i}^i\}$. Next find all slopes, say $\alpha_1^{m+1}, \dots, \alpha_{N_{m+1}}^{m+1}$, on $\partial M(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m)$, for each set of slopes $\alpha_{j_1}^1, \dots, \alpha_{j_m}^m$ with $\alpha_{j_i}^i \in \{\alpha_1^i, \dots, \alpha_{N_i}^i\}$ and having $L(\alpha_j^{m+1}) \leq C_m$, where $L(\alpha_j^{m+1})$ is the length of a normal representative of the slope α_j^{m+1} .

Construct the triangulated Dehn fillings $(M(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m, \alpha_{j_{m+1}}^{m+1}), \mathcal{T}(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m, \alpha_{j_{m+1}}^{m+1}))$ for some set of slopes $\alpha_{j_1}^1, \dots, \alpha_{j_{m+1}}^{m+1}$ with $\alpha_{j_i}^i \in \{\alpha_1^i, \dots, \alpha_{N_i}^i\}$, all i , $1 \leq i \leq$

$m + 1$. Having these triangulated Dehn fillings, we find for each their fundamental surfaces. We define \mathcal{F}^{m+1} so that $G_s^{m+1} \in \mathcal{F}^{m+1}$ if and only if there is a fundamental surface \widehat{G}_s^{m+1} in one of our Dehn fillings $(M(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m, \alpha_{j_{m+1}}^{m+1}), \mathcal{T}(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m, \alpha_{j_{m+1}}^{m+1}))$ and $\widehat{G}_s^{m+1} = G_s^{m+1}(\alpha_{j_1}^1, \dots, \alpha_{j_m}^m, \alpha_{j_{m+1}}^{m+1})$, where G_s^{m+1} is normal in M .

We have $C'_m \leq C_m$; and therefore, the slope α_j^{m+1} above is a slope α_j^{m+1} for some j and the surfaces $\{H_r^{m+1}\}$ considered above are among the surfaces $\{G_s^{m+1}\}$.

This defines the finite family \mathcal{F}^{m+1} of normal surfaces in M . We can recognize any of these surfaces that are planar and essential in M . Again, if P were to exist and if any of the surfaces in \mathcal{F}^{m+1} with projective class in $\mathcal{C}(P)$ were planar, then we would find an essential planar surface in the collection \mathcal{F}^{m+1} .

If there is a properly embedded, essential, planar surface in M , then we will find one in one of the collections $\mathcal{F}^0, \dots, \mathcal{F}^{n-2}$ or, by Corollary 4.5, we must find one in the collection \mathcal{F}^{n-1} , since each $M(\alpha_{j_1}^1, \dots, \alpha_{j_{n-1}}^{n-1})$ is a knot-manifold. If we do not find an essential, planar surface in any of the collections $\mathcal{F}^i, 0 \leq i \leq (n-1)$, then there is no properly embedded, essential, planar surface in M . \square

Just a final remark, which was alluded to at various points throughout the proof. We commented earlier that in writing a planar normal surface P as a sum of normal surfaces, F_1, \dots, F_k , then it is quite possible that none of the F_i are planar. However, what the argument above accomplishes is that we keep re-writing P as a geometric sum of normal surfaces that agree with the slope of the curves in ∂P on more and more of the boundary tori in M , while each new summand still projects into the carrier of P in $\mathcal{P}(M, \mathcal{T})$. Finally, at least if we finally get to a knot manifold, we can re-write P as a sum where for b_i the number of boundary components of F_i and b the number of boundary components of P , we have $b = \sum n_i b_i$, “the boundary of the sum is the sum of the boundaries”. In this situation, if P is planar, then one of the F_i must also be planar. Such a planar F_i is essential because it projects into $\mathcal{C}(P)$.

5. FINDING PLANAR SURFACES WITH BOUNDARY CONDITIONS

As mentioned earlier, our original interest in studying planar surfaces in knot- and link-manifolds was an interest in singular planar surfaces and, specifically, their application to problems like the Word Problem and Conjugacy Problem. These problems have boundary conditions. Hence, in hope of better understanding the problem for singular surfaces, our considerations evolved to analogous questions about embedded, planar surfaces. There are four such questions, with varying types of boundary conditions.

Recall that if μ and λ are slopes in a torus, we use $\langle \mu, \lambda \rangle$ to denote their distance; and if μ is a given slope and λ is a slope with $\langle \mu, \lambda \rangle = 1$, we call λ a *longitude with respect to μ* . Sometimes when we want to distinguish a particular slope in a component of the boundary of a given knot- or link-manifold, we call it a meridian. In particular, this is the case when we are making statements analogous to the singular problems related to the Word or Conjugacy Problems.

Suppose we are given a link-manifold M and a component B of ∂M . The various boundary conditions we consider are:

1. **(Condition B)** Can it be decided if there is a properly embedded, essential, planar surface in M with boundary meeting B ?

2. **(Condition B_γ)** Given a slope γ in B , can it be decided if there is a properly embedded, essential, planar surface in M meeting B in the slope γ ?
3. **(Condition E_γ)** Given a slope γ in B , can it be decided if there is a properly embedded, essential *punctured-disk* in M with boundary slope γ and punctures in $\partial M \setminus B$?
4. **(Condition E)** Given a meridional slope μ in B , can it be decided if there is a properly embedded, essential *punctured-disk* with boundary a longitude in B and punctures in $\partial M \setminus B$?

We show there are algorithms for Condition E_γ and Condition E ; however, we are not able to answer the question completely for either Condition B or Condition B_γ , except in the case of knot-manifolds. Condition B seems, on first glance, much like the question of the previous section. However, a closer look shows that in the previous section, we might have an essential planar surface with its boundary in B but the planar surface produced from it, which had to be in our collection of constructible surfaces to check, did not have its boundary in B . There are examples that clearly expose the deficiency of the method in the previous section. Namely, the planar surface P , with boundary meeting B , is a geometric sum of a collection of planar surfaces none of which have boundary in B and a higher genus surface with boundary in B . Under geometric sum, the handles for the higher genus surface are replaced by copies of the planar surfaces, reducing genus but adding boundary to get P . Such a sum could be quite complicated and it is not clear, if this were the case, how to find P .

However, without modification of the proof of Theorem 4.6, we have the following theorem.

5.1. Theorem. *Given a link-manifold M and a component B of ∂M , there is an algorithm to decide if there is an embedded, essential, planar surface either with boundary meeting B or with no boundary meeting B ; or neither of these possibilities occur. If a planar surface exists, the algorithm will construct one.*

This result, of course, could have been stated earlier and then we would have Theorem 4.6 as its corollary.

If we are also given a slope γ on B , as in Condition B_γ , then, again, without modification of the proof of Theorem 4.6 and using the notion given below on normal surfaces with boundary conditions, the conclusion of Theorem 5.1 can be replaced with:

there is an algorithm to decide if there is an embedded, essential planar surface either with slope γ on B or with no boundary on B ; or neither of these possibilities occur. If a planar surface exists, the algorithm will construct one.

5.1. Boundary conditions on normal surfaces. The remaining problems we investigate in this section search for normal surfaces with a given boundary slope; this is the prototypical example of a boundary condition on a normal surface.

In what follows we assume the reader is familiar with normal curves in triangulated 2-manifolds.

Suppose M is a 3-manifold and \mathcal{T} is a triangulation of M . Select some order for the normal triangle and quadrilateral types in the tetrahedra of \mathcal{T} , which then determines coordinates in \mathbb{R}^{7t} ; similarly, select some order for the normal arc types in ∂M , which then determines coordinates in \mathbb{R}^m , where m is the number of arc

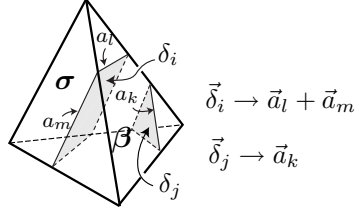


FIGURE 5. Normal boundary operator in the case that the faces σ and β are in ∂M .

types in the triangulation induced on ∂M . Now, if δ_i is a normal triangle or quad type in M , we associate with δ_i the unit vector $\vec{\delta}_i = (x_1, \dots, x_q, \dots, x_{7t})$, where $x_q = 0, q \neq i, x_i = 1$; similarly, if a_k is a normal arc type in ∂M , we associate the unit vector $\vec{a}_k = (z_1, \dots, z_p, \dots, z_m)$, where $z_p = 0, p \neq k, z_k = 1$. There is a *normal boundary operator*, denoted $\partial_{\mathcal{T}}$, which is a linear map from \mathbb{R}^{7t} to \mathbb{R}^m defined as

$$\partial_{\mathcal{T}}(\vec{\delta}_i) = \sum \epsilon_{i,j} \vec{a}_j,$$

where $\epsilon_{i,j} = 1$ if the arc type a_j in ∂M is in the normal disk type δ_i and $\epsilon_{i,j} = 0$, otherwise. See Figure 5.

Now, suppose M is a link-manifold, B is a boundary component and \mathcal{T} is a triangulation of M that induces a one vertex triangulation on B . From [11], we have that the slopes on B are in one-one correspondence with normal isotopy classes of nontrivial simple closed curves in B . In the above scheme, it follows that if γ is a slope, then γ corresponds to a unique integer lattice point in the parametrization space, \mathbb{R}^m , of normal curves in ∂M and thus determines a unique linear subspace, say $R(\gamma)$. The inverse of $R(\gamma)$ under the boundary operator $\partial_{\mathcal{T}}$ is a vector subspace of \mathbb{R}^{7t} and meets the normal solution space, $\mathcal{S}(M, \mathcal{T})$, for normal surfaces in M in a sub-cone. This sub-cone, contains the parametrization of those normal surfaces in M that either meet B only in the slope γ or do not meet B at all (which includes the closed normal surfaces). We denote this sub-cone $\mathcal{S}_{\gamma}(M, \mathcal{T})$. It satisfies the algebra we developed for the normal solution space, such as having a projective solution space, which will be a subcell of the projective space for M , having vertex and fundamental solutions associate with it (however, these are not necessarily vertex and fundamental solutions of $\mathcal{S}(M, \mathcal{T})$), and so on.

While the above description of $\mathcal{S}_{\gamma}(M, \mathcal{T})$ provides a nice theoretical setting, there is a direct method of determining $\mathcal{S}_{\gamma}(M, \mathcal{T})$ using a system of homogeneous linear equations. To see this recall from [11] that since \mathcal{T} induces a one-vertex triangulation on B , the normal isotopy classes of normal curves in B can be parameterized by just three normal arc types in one of the two triangles in the induced triangulation on B (m above can be taken to be 3). See Figure 6. Choose notation so that the three arc types are a_1, a_2 and a_3 . If we consider only essential curves in B and prescribe a slope γ , then we have that one of the coordinates must be zero (otherwise, we would allow trivial curves) and the other two must be in constant ratio. In particular, if $z_k, k = 1, 2, 3$, is the number of arcs of type a_k in γ and, say, $a_2 = 0$, then z_1, z_3 are in a constant ratio; i.e., there are relatively prime integers r, s (possibly $r = 1 = s$) so that $z_1/z_3 = r/s$. Denote the normal triangles that meet B in the arc type a_k by δ_{i_k} and the normal quadrilaterals that meet B in

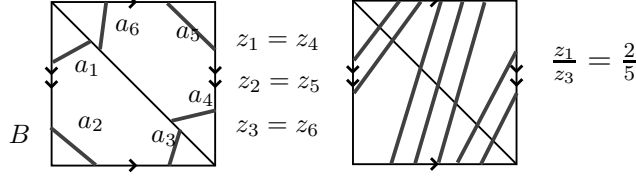


FIGURE 6. Normal arcs in the one-vertex triangulation of the torus, the equations for normal curves, and an example of the slope corresponding to the ratio $z_1/z_3 = 2/5$.

the arc type a_k by $\delta_{j_k}^Q$. Then if we add to the matching equations for (M, T) the equations

$$x_{i_2} + y_{j_2} = 0$$

and

$$s(x_{i_1} + y_{j_1}) = r(x_{i_3} + y_{j_3}),$$

we recover $\mathcal{S}_\gamma(M, T)$ as the solution cone in the positive orthant of \mathbb{R}^{7t} for this system of equations.

5.2. Boundary conditions and average length estimate. Suppose M is a link-manifold, B is a component of ∂M , and γ is a slope on B . Now, suppose F is an embedded normal surface and b_γ is the number of components of ∂F having slope γ . Let $b_{-\gamma}$ denote the number of components of ∂F that do not have slope γ ; if b is the number of components of ∂F , $b = b_\gamma + b_{-\gamma}$. If $b_{-\gamma} \neq 0$, we set $\lambda_{av}^{-\gamma} = [(L(\partial F) - b_\gamma L(\gamma))/b_{-\gamma}]$ and call $\lambda_{av}^{-\gamma}$ the *average length of ∂F away from γ* . If F does not meet B in slope γ , then this is just the average length of ∂F . We have the following modification of Proposition 3.2

5.2. Proposition. *Suppose M is a link-manifold with no embedded annuli having boundary in distinct components of ∂M , B is a component of ∂M , and γ is a slope on B . Furthermore, suppose T is a 0-efficient triangulation of M . Then there is a constant $C = C(M, T)$, depending only on M and T so that if F is an embedded normal surface in M with no trivial boundary curves and $\lambda_{av}^{-\gamma}$ is the average length of the components of ∂F away from γ , then*

$$\lambda_{av}^{-\gamma} \leq C(2g + 2 + b_\gamma),$$

where g is the genus of F and b_γ bounds the number of components of ∂F having slope γ .

Proof. We use the same notation and proceed just as in the proof of Proposition 3.2. We define

$$C = \max \left\{ \frac{L(\partial F_1)}{-\chi(F_1)}, \dots, \frac{L(\partial F_I)}{-\chi(F_I)}, \frac{L(\partial A_1)}{|A_1|}, \dots, \frac{L(\partial A_K)}{|A_K|} \right\}.$$

We have b_γ bounding the number of components of ∂F having slope γ ; $b_{-\gamma}$ the number of components of ∂F not having slope γ ; $b_A = \sum n_k |A_k|$, the number of boundary components of $\sum n_k A_k$; and b the number of boundary components of F . It follows that

$$b_{-\gamma} \lambda_{av}^{-\gamma} \leq b_{-\gamma} \lambda_{av}^{-\gamma} + (b - b_{-\gamma}) L(\gamma) = L(\partial F) = \sum l_i L(\partial F_i) + \sum n_k L(\partial A_k) \leq$$

$$\leq C \left(\sum l_i(-\chi(F_i)) + \sum n_k |\partial A_k| \right) = C(-\chi(F') + b_A).$$

However, $\chi(F) = \chi(F')$; hence, we have $b_{-\gamma} \lambda_{av}^{-\gamma} \leq C(2g - 2 + b + b_A)$. It follows that

$$\lambda_{av}^{-\gamma} \leq C \left(\frac{2g-2}{b_{-\gamma}} + \frac{b_{\gamma}}{b_{-\gamma}} + 1 + \frac{b_A}{b_{-\gamma}} \right) \leq C \left(2g + \frac{b}{b_{-\gamma}} + 1 \right) \leq C(2g + 2 + b_{\gamma}).$$

□

5.3. Conditions E_{γ} and E . We first construct an algorithm for Condition E_{γ} , using induction on the number of boundary components of the given link-manifold. Afterwards, given a link-manifold, we are able to derive an algorithm for Condition E by using the algorithm for Condition E_{γ} in a related link-manifold, which is constructed from the given link-manifold. For our induction step in the algorithm for E_{γ} , we use Dehn filling to reduce the number of boundary components and invoke the induction hypothesis. However, we essentially go the opposite direction in deriving the algorithm for Condition E ; in this case, we construct a new link-manifold by removing a knot from the given link-manifold, a process which we might think of as “Dehn drilling”.

Suppose D is a planar surface with three boundary components (a pair-of-pants). We wish to understand the embedded, essential planar surfaces in $\eta = D \times S^1$. Suppose b_1, b_2 and b_3 are the boundary components of D and $B_i = b_i \times S^1, i = 1, 2, 3$, denotes the corresponding components of $\partial\eta$. For $t \in S^1$, we call the slope determined by $t \times b_i$ meridional on B_i and denote it by μ_i ; for any point $x \in b_i$, we call the slope determined by $x \times S^1$ vertical and denote it by α_i . The product structure on η makes it a Seifert fiber space. See Figure 7.

Just as we distinguished a special boundary component of a planar surface in our definition of a punctured-disk, if we have a planar surface with at least two boundary components and we distinguish two distinct boundary components, we call the planar surface a *punctured-annulus* and refer to all boundary components distinct from the two we have distinguished as the punctures. Again, see Figure 7. If P is a punctured-annulus, C' and C'' the two boundary components we have distinguished, then we set $bd(P) = C' \cup C''$ and call $bd(P)$ the boundary of P .

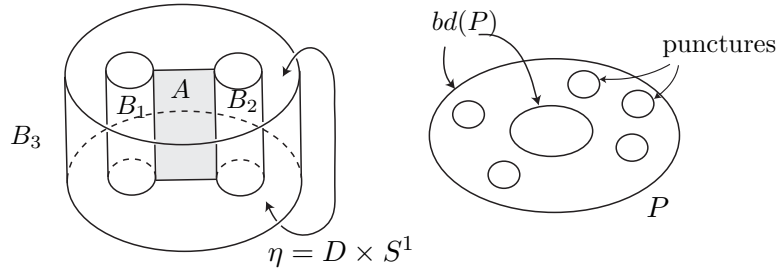


FIGURE 7. A pair-of-pants crossed with the circle, η , which is homeomorphic with the neighborhood $\eta(A)$ of $B' \cup A \cup B''$ where A is an essential annulus in a link-manifold having boundary in distinct boundary components $B'(B_1)$ and $B''(B_2)$. On the right is a punctured-annulus P with boundary $bd(P)$.

For each essential arc a in D , we get an embedded, essential, vertical annulus $a \times S^1$ in η . There are three that have their boundaries in distinct components of $\partial\eta$; we denote these $A_{1,2}$, $A_{1,3}$ and $A_{2,3}$, where $A_{i,j}$ is a vertical annulus having one boundary, α_i in B_i and the other α_j in B_j .

Since there is a symmetry between the boundary components of η , we shall distinguish one, say B_2 , and describe a family of embedded, essential planar surfaces in η . Each of the surfaces we describe is a punctured-annulus having one boundary in B_1 , the other in B_3 , and all its punctures in B_2 . We get an analogous family by distinguishing either B_1 or B_3 . Notice that a Dehn filling of η along the slope μ_2 gives the manifold $\eta(\mu_2) = (S^1 \times S^1) \times [0, 1]$, a torus cross the interval, with $B_1 = S^1 \times S^1 \times \{1\}$ and $B_3 = S^1 \times S^1 \times \{0\}$. For every slope γ_1 in B_1 there is an annulus $\gamma_1 \times [0, 1]$ in $\eta(\mu_2)$ meeting B_1 in γ_1 and B_3 in a corresponding slope, say γ_3 ; and from this annulus, we obtain a punctured-annulus in η , having all its punctures in B_2 in the slope μ_2 . We denote this punctured-annulus $P_{1,3}^\gamma$. Now, having the various vertical annuli, $A_{i,j}$, described above, we can Dehn twist the punctured-annulus $P_{1,3}^\gamma$ about $A_{i,j}$. Dehn twisting about $A_{1,3}$ gives the same punctured-annuli that we would have gotten from our construction had we first twisted the slope γ_1 about the vertical slope α_1 on B_1 . However, Dehn twisting about $A_{1,2}$ or $A_{2,3}$ gives new families of punctured-annuli. By twisting $P_{1,3}^\gamma$ about $A_{1,2}$ we get a family of punctured-annuli that meet B_1 in the family of slopes obtained from γ_1 by Dehn twisting about the vertical slope α_1 , using the same number of twist as we have about $A_{1,2}$; they meet B_3 in the slope γ_3 ; and they meet B_2 in a number of punctures, each having a slope obtained from μ_2 by Dehn twisting about α_2 , using the same number of twist as we have about $A_{1,2}$. We get an analogous family if we twist about $A_{2,3}$ except that we do not change the slope on B_1 in these cases.

5.3. Lemma. *The above examples describe all possible embedded, essential planar surfaces in $\eta = D \times S^1$.*

Proof. Suppose P is an embedded, essential planar surface with boundary in only one component of $\partial\eta$, say B_1 . Then by Dehn filling along the meridian slopes, μ_2 and μ_3 , we get a solid torus, $\mathbb{T} = \eta(\mu_2, \mu_3)$. The only planar surfaces in \mathbb{T} meeting $B_1 = \partial\mathbb{T}$ in essential curves are the meridional disk and an annulus parallel into B_1 . But P having boundary only in B_1 leaves only the possibility that P is a vertical annulus; furthermore, to be essential in η , P must separate the boundary components, B_2 and B_3 .

If P meets precisely two of the boundary components, say B_1 and B_2 , then we can Dehn fill η along μ_3 . The Dehn filling $\eta(\mu_3)$ is an annulus cross the circle (torus cross an interval) and all embedded, essential planar surfaces with their boundaries in distinct components of the boundary are annuli. For such a surface not to meet the third boundary component of η , it must be vertical in η .

The only remaining possibility is that P meets all three boundary components of η . In this case, we note that P can not meet a boundary component in its vertical slope. If P did meet, say B_3 in α_3 , then we can Dehn fill η along α_3 , getting the essential planar surface $P(\alpha_3)$ obtained by capping off P in $\eta(\alpha_3)$, which is the connected sum of two solid tori. But since $P(\alpha_3)$ has boundary in each boundary component of $\eta(\alpha_3)$, P could not be essential in η .

It follows that after Dehn filling η along all the boundary slopes of P , the Seifert fiber structure on η extends to a Seifert fiber structure on the Dehn filled manifold;

however, capping off P gives an embedded, horizontal 2-sphere. It follows that one of the slopes of ∂P on the boundary of η is meridional (possibly after a Dehn twist about a vertical annulus). Thus P is a punctured-annulus with its punctures on this boundary component of η that P meets in meridional slopes. Reversing the Dehn twist, gives us one of the punctured-annuli described above. \square

Splitting a manifold along a properly embedded surface is a standard notion; however, we want a special form of this when splitting a link-manifold along a properly embedded annulus having its boundary in distinct components of the boundary of the link-manifold. Suppose M is a link-manifold with B' and B'' distinct boundary components of ∂M ; and suppose A is an embedded, essential annulus with one boundary in B' and one boundary in B'' . Denote a small regular neighborhood of $B' \cup A \cup B''$ by $\eta(A)$. Then $\eta(A)$ is a disk with two punctures (a pair of pants) crossed with the circle; see Figure 7. If we denote the frontier of $\eta(A)$ by B_A , then the boundary components of $\eta(A)$ are B', B'' , and B_A . Let $M_A = M \setminus \overset{\circ}{\eta}(A)$, then M_A is a link-manifold with a boundary component B_A and one fewer boundary component than the link-manifold M . We say M_A is obtained from M by *splitting M along A* . Finally, since $\eta(A)$ is a pair-of-pants cross S^1 , a slope on any one of the three boundary components can be uniquely associated with a "parallel" slope on the other two boundary components; we shall use the convention in this situation of γ' being the slope on B' , γ'' on B'' and γ_A on B_A (if $B' = B$, then we use γ for γ'). We refer to the meridional slopes designated above for η as meridional slopes on B' (B), B'' or B_A in $\eta(A)$ and write μ_A for B_A , and so on. In Lemma 5.3, we characterized embedded, essential planar surfaces in $\eta(A)$. Now, we provide the relationship between embedded, essential planar surfaces in M and those in M_A .

5.4. Proposition. *Suppose M is a link-manifold, B is a component of ∂M and A is an embedded, essential annulus having its boundary in distinct components B' and B'' of ∂M . Let M_A be the link-manifold obtained by splitting M along A . We have the following:*

- (1) *For $B' \neq B \neq B''$.*
 - (a) *There is an embedded, essential planar surface in M with boundary meeting B if and only if there is one in M_A with boundary meeting B .*
 - (b) *There is an embedded punctured-disk in M with its boundary meeting B in slope γ if and only if there is one in M_A with its boundary meeting B in slope γ .*
- (2) *For $B' = B \neq B''$.*
 - (a) *Either the only embedded, essential planar surfaces meeting B are annuli meeting in slope α and there are no embedded, essential planar surfaces in M_A meeting B_A or there is an embedded, essential planar surface in M with boundary meeting B if and only if there is one in M_A with boundary meeting B_A .*
 - (b) *There are embedded punctured-disks in M with boundary meeting B in every slope if and only if there is an embedded punctured-disk in M_A with boundary meeting B_A in a slope obtained from μ_A by Dehn twisting about α_A (i.e., a slope having geometric intersection one with α_A).*

- (c) *If the previous situation does not hold, either the only embedded punctured-disk in M meeting B is A and there are no embedded punctured-disks in M_A meeting B_A or there is an embedded punctured-disk in M with boundary meeting B in slope γ if and only if there is an embedded punctured-disk in M_A with boundary meeting B_A in a slope obtained from γ_A by Dehn twisting about α_A .*

Proof. Proof for 1(a) and (b). Suppose P is an embedded, essential planar surface in M and its boundary meets B . We isotope P so that it meets B_A minimally and transversely; this does not affect any components of ∂P in B . Hence, if P meets $\eta(A)$ at all, it must meet $\eta(A)$ in embedded, essential planar surfaces in $\eta(A)$. It follows from Lemma 5.3 that removing such pieces from P leaves a connected embedded, essential planar surface in M_A meeting B exactly as P did.

Conversely, suppose P_A is an embedded, essential planar surface in M_A and it meets B . If it does not meet B_A , it is also an embedded, essential planar surface in M . If it does meet B_A , then we consider the slope of its boundary in B_A . By Lemma 5.3, no matter the slope there is an embedded, essential punctured annulus in η_A having just one of its boundary components in B_A and having this slope. If P_A meets B_A in m components, we add m copies of such a punctured-annulus in η_A , arriving at the desired planar surface for M . Notice that this also proves part 1(b) as well, since none of this had any affect on the meets with B .

Proof of 2(a). Here we might have an essential, planar surface in M meeting B but there are none in M_A meeting B_A ; that is, the planar surface in M is contained in $\eta(A)$. However, from Lemma 5.3, we have that such a surface is then one of the vertical annuli in $\eta(A)$. Otherwise, there is an embedded essential surface in M_A meeting B_A if and only if there is one in M meeting B .

Proof of 2(b) and (c). Suppose we have an embedded punctured-disk in M with its boundary in B having slope γ . Then by an isotopy, we may make it meet B_A transversely and minimally. Thus we have that a component of its intersection with $\eta(A)$ is a punctured-disk in $\eta(A)$ with its boundary having slope γ in B (if it has more than one component in $\eta(A)$, then each must be a vertical annulus and have at least one boundary component in B_A ; in which case $\gamma = \alpha$). We use the characterization of planar surfaces in $\eta(A)$ given in Lemma 5.3 to determine the possibilities.

If our punctured-disk meets $\eta(A)$ only in vertical annuli, then $\gamma = \alpha$ and we have the possibility that there are no punctured-disk in M_A having boundary in B_A or there is a punctured-disk in M_A having its boundary of slope α_A in B_A . This satisfies the conclusion to part 2(c). Notice that if there are no punctured-disk in M_A meeting B_A in α_A , there could be an embedded, essential planar surface in M_A having several boundary components with slope α_A on B_A and it could be extended to an embedded punctured-disk in M ; but its boundary slope on B would be α and we would not get a new slope.

So, suppose our embedded punctured-disk in M does not meet $\eta(A)$ in vertical annuli. Hence, there is just one component, which by Lemma 5.3 is a punctured-annulus having one of its boundary components in B with slope γ .

We consider the two possibilities as to where the other boundary component of this punctured-annulus is; it is either in B'' or in B_A .

If the other boundary component is in B'' , then the punctured-annulus meets B_A in a slope, say μ'_A that can be Dehn twisted about α_A to μ_A , the meridian on

B_A . Furthermore, μ'_A must bound a punctured-disk in M_A . Thus given any slope β on B , we can construct a punctured-annulus in $\eta(A)$ having one boundary on B with slope β , the other boundary on B'' , and punctures in μ_A . We can then Dehn twist this punctured-annulus about an essential annulus in $\eta(A)$ between B'' and B_A that has slope α'' on B'' and slope α_A on B_A . This gives a punctured-annulus in $\eta(A)$ meeting B in slope β and punctures in B_A all having slope μ'_A . It follows that β bounds a punctured-disk in M . This gives the conclusion for part 2(b).

Finally, we assume our embedded punctured-disk in M does not meet $\eta(A)$ in a vertical annulus and there are no embedded punctured-disks in M_A meeting B_A in a slope obtained by twisting μ_A about α_A .

Now, by Lemma 5.3, we have that our punctured-disk must meet $\eta(A)$ in a punctured-annulus having one boundary the slope γ in B , the other boundary in B_A and its punctures in B'' . Hence, the slope on B_A can be any obtained from γ_A by a Dehn twist about α_A .

Clearly, if we have a punctured-disk in M_A with boundary in B_A , we can use Lemma 5.3 to construct a punctured-disk in M with boundary in B and the slopes will satisfy the relationship of our conclusions in our lemma. \square

Remark 5.1. From Part 2(c) of Theorem 5.4 we see that if there is an essential annulus A between B and B'' , then for every slope γ_A in B_A that bounds a punctured-disk in M , with punctures in $\partial M_A \setminus B_A$, there is generated an infinite family of punctured-disks in M each with its boundary in B having slopes corresponding to Dehn twisting about A .

-Algorithm for Condition E_γ . Our proof in the case of Condition E_γ is by induction on the number n of boundary components of the link-manifold M . We note that in the proof of Theorem 4.6 in the previous section, we did not use induction but we did use Dehn fillings to regularly reduce the number of boundary components. We could not use induction in the previous section because we may, after filling, no longer have an essential planar surface. Now, for an embedded punctured-disk, as in Conditions E_γ and E , the existence of an embedded punctured-disk assure the existence of an essential one.

5.5. Lemma. *Suppose M is an irreducible link-manifold, B a component of ∂M and γ a slope in B . If M contains an embedded punctured-disk with boundary having slope γ and punctures in $M \setminus B$, then M contains an embedded, essential punctured-disk with boundary having slope γ and punctures in $\partial M \setminus B$.*

Proof. We have observed that a properly embedded surface in a link-manifold is essential if and only if it is incompressible and not an annulus or torus parallel into the boundary. Thus suppose P is a properly embedded punctured-disk with $bd(P)$ having slope γ in B and all punctures in $\partial M \setminus B$. If P is incompressible, then since it has precisely one boundary component in B , it must be essential; however, if P were not incompressible, a compressions on P leaves an embedded planar surface with precisely one component of its boundary ($bd(P)$) in B and all other punctures in $\partial M \setminus B$. After a finite number of compressions, we get the desired, essential punctured-disk. \square

5.6. Theorem. (Condition E_γ) *Given a link-manifold M , a component B of ∂M , and a slope γ in B , there is an algorithm to decide if M contains an embedded,*

essential punctured-disk with boundary having slope γ and punctures in $\partial M \setminus B$. If there is one, the algorithm will construct one.

Proof. Our proof is by induction on the number n of boundary components of the given link-manifold. We begin with one boundary component.

($n = 1$) M is a knot-manifold.

Suppose the knot manifold M is given by the triangulation \mathcal{T} . For a knot manifold Condition E_γ is the question of whether there is an embedded disk with boundary slope γ . Consider the normal solution space $\mathcal{S}_\gamma(M, \mathcal{T})$ of normal surfaces satisfying the boundary condition of meeting ∂M in the slope γ . Compute the fundamental surfaces for M in $\mathcal{S}_\gamma(M, \mathcal{T})$. From Lemma 2.6, there is an embedded, essential disk with boundary having slope γ in M if and only if there is one among these fundamental surfaces.

($n \Rightarrow n + 1$) Our induction hypothesis is that given a link-manifold M , a component B of ∂M , and a slope γ on B , then if M has no more than $n, n \geq 1$, boundary components, we can decide if there is an embedded, essential punctured-disk in M with boundary the slope γ and punctures in $\partial M \setminus B$. Furthermore, if there is one, the algorithm will construct one.

So suppose we are given a link-manifold M with $n + 1$ boundary components; B is a selected boundary component of ∂M ; and γ is a slope on B .

By Theorem 4.1, we can construct a prime decomposition of $M = p(S^2 \times S^1) \# q(\mathbb{R}P^3) \# r(D^2 \times S^1) \# M_1 \# \cdots \# M_K$, where each M_i is given by a 0-efficient triangulation. In the construction of such a prime decomposition, if B results in being a boundary component of some solid torus in the decomposition, then the algorithm actually constructs an embedded, essential disk with boundary in B . From Proposition 4.2 and the fact that there is a unique slope in the boundary of a solid torus bounding an embedded, essential disk, we can decide Condition E_γ . Furthermore, if B is in a prime factor having fewer boundary components than M , then we can invoke our induction hypothesis.

Hence, we may assume M has $n + 1$ boundary components and is given by a 0-efficient triangulation \mathcal{T} . It follows that M has no normal 2-spheres, has no embedded $\mathbb{R}P^2$, the only normal disks are vertex-linking; hence, M is irreducible and ∂ -irreducible. Furthermore, all vertices of \mathcal{T} are in ∂M and each component of ∂M has precisely one vertex. We consider the solution space $\mathcal{S}_\gamma(M, \mathcal{T})$ and compute its fundamental surfaces.

Case A. Suppose there is a fundamental surface that is an essential annulus A having boundary in distinct components $B' \neq B''$ of ∂M .

If $B' = B \neq B''$, then A is the desired punctured-disk having boundary γ and one puncture in B'' .

If $B' \neq B \neq B''$, then split the link-manifold M along A . We get a new link-manifold M_A . By part 1(b) of Lemma 5.4 there is an embedded punctured-disk in M having boundary slope γ in B if and only if there is one in M_A . The link-manifold M_A has n boundary components. By our induction hypothesis, we can decide if M_A contains an embedded punctured-disk with boundary having slope γ in B . If we find one in M_A , then Lemma 5.4 tells us how to construct one in M .

We remark that M_A is not given by a triangulation but a cell-decomposition. However, from this cell-decomposition, we can construct a link-manifold $N \subset M_A$ (but not necessarily M_A) given by a triangulation, typically with fewer tetrahedra

than \mathcal{T} ; N has no more than n boundary components, one of which is B , and has the property that there is an embedded punctured-disk in N with boundary slope γ in B if and only if there is an embedded punctured-disk with boundary slope γ in M_A with boundary slope γ .

Finally, we are left with the situation that M is given by a 0-efficient triangulation \mathcal{T} and there are no embedded, essential annuli having their boundary in distinct components of ∂M and are also represented in the solution space $\mathcal{S}_\gamma(M, \mathcal{T})$.

If there is any embedded punctured-disk in M with boundary having slope γ , say P , then, having the hypothesis of Proposition 5.2, there exists a constant C , depending only on M and \mathcal{T} , so that the average length of the punctures in P is no larger than $3C$, $\lambda_{av}^- \leq 3C$.

Construct the set $\{\alpha_1, \dots, \alpha_K\}$ of all slopes on the components of $\partial M \setminus B$ that have the property $L(\alpha_i) \leq C$. Now, construct the triangulated Dehn fillings, $M(\alpha_1), \dots, M(\alpha_K)$. There is a punctured-disk in M with boundary having slope γ and punctures in $\partial M \setminus B$ if and only if there is an embedded punctured-disk in some $M(\alpha_i)$ with boundary having slope γ and punctures in $\partial M(\alpha_i) \setminus B$. But each $M(\alpha_i)$ is a link-manifold with no more than n boundary components. By our induction hypothesis, we can decide if there is such an embedded punctured-disk in some $M(\alpha_i)$. If there is one, the algorithm will construct one; the component of such a punctured-disk that meets B after removing the interior of the layered solid torus $\mathbb{T}(\alpha_i)$ is a desired solution for the given manifold M . \square

-Algorithm for Condition E. Recall that if we designate a slope μ on the boundary of a link-manifold, we often call it a meridian and any slope λ with $\langle \mu, \lambda \rangle = 1$ is called a longitude (with respect to μ). Condition E ask if given a meridian, can we decide if a longitude bounds a punctured-disk. *A priori* this seems much more daunting than answering Condition E_γ , as we are asking here if some slope among an infinite family of slopes bounds a punctured-disk. We are able to solve this problem quite easily and, in fact, use Condition E_γ to answer Condition E . We first need an observation.

Suppose M is a link-manifold, B is a component of ∂M , and μ is a slope in B . Let $\eta(B) = B \times [0, 1]$ be a small product neighborhood of the boundary component B and choose notation so $B = B \times 1$. Let $\mu_0 = \mu \times \{0\}$ (here we pick any representative for μ). If we remove the interior of a small tubular neighborhood of μ_0 , we have a link-manifold with one more boundary component than M . We use M^μ to denote this new link-manifold and use B^μ to designate the boundary component of M^μ that is not a component of ∂M . The unique slope on B^μ that bounds a disk in the tubular neighborhood of μ_0 is called the meridional slope on B^μ and designated by μ^* ; it is a meridian in the typical sense as it bounds the meridional disk of the tube about μ_0 . The embedded, essential annulus $A = \mu \times [0, 1] \cap M^\mu$ has boundary slope μ in B and boundary slope a longitude, say λ^* (with respect to μ^*), in B^μ . We say M^μ is the link-manifold obtained from M by Dehn drilling along the slope μ .

5.7. Proposition. *Let M be a link-manifold, B a component of ∂M , and μ a slope in B . There is an embedded punctured-disk in M with boundary the slope of a longitude in B if and only if every longitude in B bounds a punctured-disk in M^μ .*

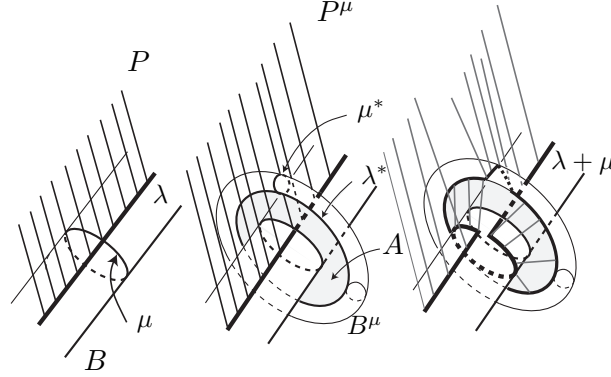


FIGURE 8. Dehn twisting a punctured-disk with boundary a longitude in B about the annulus A in the link-manifold M^μ obtained by Dehn drilling along the slope μ .

Proof. Suppose there is a punctured-disk P in M having boundary a longitude in B . If $P^\mu = P \cap M^\mu$, then (possibly after an isotopy) we have P^μ meeting B^μ in the meridional slope μ^* . See Figure 8. It follows that Dehn twisting P^μ about the annulus A gives an embedded punctured-disk in M^μ with boundary a longitude in B ; however, every longitude in B can be realized this way.

Conversely, if there is an embedded punctured-disk in M^μ with boundary a longitude in B , then there is one for every longitude and, in particular, after an appropriate Dehn twist about A , there will be one that meets B^μ in a single meridian. Such a punctured-disk gives the desired punctured-disk in M by Dehn filling M^μ along the meridian in B_μ to reclaim M . \square

5.8. Theorem. (Condition E) *Given a link-manifold M , a component B of ∂M , and a slope μ in B , there is an algorithm to decide if M contains an embedded, essential punctured-disk with boundary having slope of a longitude in B (with respect to μ) and punctures in $\partial M \setminus B$. If there is one, the algorithm will construct one.*

Proof. This follows quite straight forward from Proposition 5.7. We construct the manifold M^μ obtained from M by Dehn drilling along μ . Select any longitude λ in B and run the algorithm Condition E_λ . From Proposition 5.7, there is an embedded punctured-disk in M having slope a longitude (not necessarily λ) and punctures in $M \setminus B$ if and only if there is an embedded punctured-disk in M^μ with boundary having slope λ and punctures in $\partial M^\mu \setminus B$. \square

Remark 5.2. We note that we have not said anything about a triangulation of M^μ . It may be the case that the slope μ is quite long relative to the given triangulation on M , which indicates that we may need to add a number of tetrahedra; also, we must add boundary and at least one more vertex. There is, however, a nice way to triangulate M^μ that addresses these suspected needs in a very visible way.

Suppose M is given by a triangulation \mathcal{T} . Using the notation from above, we see that a small neighborhood of $B \cup A \cup B^\mu$ is a pair of pants crossed with a circle. We shall add a “pinched” pair of pants crossed with the circle to M along B in such a way as to get the manifold M^μ and extend \mathcal{T} to a triangulation \mathcal{T}^μ of the Dehn drilling M^μ . See Figure 9.

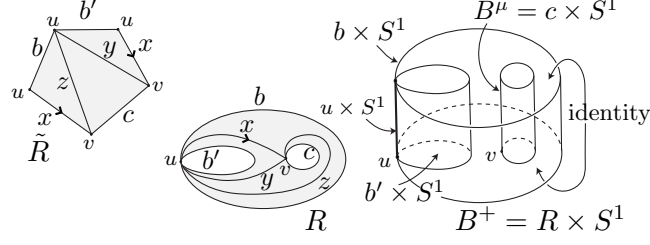


FIGURE 9. A modification to the given triangulation for M to a triangulation of the manifold M^μ obtained by Dehn drilling along the slope μ in B .

Let \tilde{R} be a pentagon in the plane with two boundary edges labeled x to be identified as indicated, three vertices labeled u to be identified, and two vertices labeled v to be identified. Label the three remaining boundary edges b, b' and c , as shown. Triangulate R (any way is fine). After identification, this gives a triangulation of a “pinched” pair-of-pants, R . We take the product $R \times S^1$, denoted B^+ . We can triangulate B^+ with nine tetrahedra so that $b, b', c, u \times S^1$ and $v \times S^1$ are all edges. We wish to identify the boundary $b' \times S^1$ of B^+ with B to obtain the manifold M^μ . To do this, the slope of $u \times S^1$ must be identified with the slope μ ; then the boundary $c \times S^1$ corresponds to B^μ . We also want the identification to be simplicial so that we can extend the triangulation \mathcal{T} of M to a triangulation of M^μ .

To accomplish these requirements, we layer tetrahedra onto B , at each layering getting a new triangulation of M , and eventually arriving at a triangulation of M that has added no new vertices and has an edge, say e^μ with slope μ . We identify $b' \times S^1$ with B , taking $u \times S^1$ to e^μ and b' to either of the other edges. This determines a diagonal on the boundary triangulation of $b' \times S^1$, which can be realized by a nine tetrahedra triangulation of B^+ .

This gives a minimal vertex-triangulation of M^μ . Note that the layerings on B are determined by the length of μ , a necessity we expected, and the nine additional tetrahedra allow room for the new boundary and vertex on $c \times S^1 = B^\mu$.

5.4. The family of slopes for punctured-disks. From our methods to obtain the algorithm for Conditions E_γ , we are able, if given a link-manifold and a component B of ∂M , to give a construction that lists precisely those slopes on B that are boundary slopes of embedded punctured-disks in M .

It seems that algorithms for Condition E_γ and Condition E might follow from this result. However, the list may be infinite and we have the classical problem in obtaining algorithms. Namely, if there is an embedded punctured-disk in M satisfying either of the boundary conditions, E_γ or E , then by systematically constructing precisely those slopes in B that can bound embedded, punctured disks, we will encounter the corresponding slope and have our solution. But if no slope does satisfy Condition E_γ or Condition E and the set of possible slopes is infinite, when do we stop looking? We believe, however, that the construction is quite informative.

5.9. Theorem. *Suppose we are given a link-manifold M and a component B of ∂M . There is a constructible set of slopes \mathcal{S} in B so that*

- (i) any embedded punctured-disk in M with boundary in B and punctures in $\partial M \setminus B$ has boundary slope in \mathcal{S} ; and
- (ii) every slope in \mathcal{S} is the boundary slope of an embedded punctured-disk in M with boundary in B and punctures in $\partial M \setminus B$.

Proof. The proof of Theorem 5.9 is by induction on the number, n , of boundary components of the given link-manifold M .

($n = 1$) For a knot-manifold, an embedded punctured-disk with boundary in B and punctures in $\partial M \setminus B$ is an embedded disk. By Lemma 2.6 there is an algorithm to decide if M contains an embedded disk with its boundary in B . If it does, then M splits as a connected sum of a solid torus with boundary B and a closed 3-manifold. In this case \mathcal{S} contains a single slope. Otherwise, $\mathcal{S} = \emptyset$.

($n \Rightarrow n + 1$) Our induction step assumes that given a link-manifold M' that has no more than n boundary components, $n \geq 1$, along with a distinguished component B' of $\partial M'$, then there is a constructible set of slopes \mathcal{S}' in B' so that items (i) and (ii) are satisfied if we substitute M' , B' and \mathcal{S}' for M , B and \mathcal{S} , respectively.

Now, suppose we are given the link-manifold M that has $n + 1$ boundary components and B is a specified component of ∂M .

We use Theorem 4.1 to construct a prime decomposition of M so that B is either the boundary of a solid torus or B is the boundary of a prime factor M' given by a 0-efficient triangulation. In the former case \mathcal{S} consists of the unique single meridional slope; in the latter case, if M' has fewer than $n + 1$ boundary components, then we apply induction. Hence, we may assume M has $n + 1$ boundary components and is given by a 0-efficient triangulation. Compute the fundamental surfaces of M .

We may assume no fundamental surface is an embedded, essential disk, giving the following possibilities.

First, we consider the possibility that a fundamental surface is an embedded, essential annulus having boundary in distinct components of ∂M .

We note that for a fixed component of ∂M there is at most one slope that is in the boundary of an embedded, essential annulus; or $M = S^1 \times S^1 \times I$ (M has more than one boundary component).

Suppose there is an embedded, essential annulus A between the components B' and B'' , $B' \neq B''$ of ∂M .

There are two possibilities; either $B' \neq B \neq B''$ or $B' = B \neq B''$.

1. $B' \neq B \neq B''$. Split M at A to get the link-manifold M_A . Then M_A has fewer boundary components than M and B is a component of ∂M_A . By Propositions 5.4 Part 1(b) we have that $\mathcal{S} = \mathcal{S}_A$, where \mathcal{S}_A is the set of slopes on B , as a component of ∂M_A , that bound punctured-disks in M_A with punctures in $\partial M_A \setminus B$.

To construct the slopes in \mathcal{S} , we need to construct the slopes in \mathcal{S}_A . In this situation, we provided a method for a triangulation of M_A in the Remark following the proof of Theorem 5.6.

2. $B' = B (B \neq B'')$. Again, we split M at A to get the link-manifold M_A . Then M_A has fewer boundary components than M and B_A becomes the distinguished component of ∂M_A .

By Propositions 5.4 we can have any one of three possibilities. From Part 2(b) there are embedded punctured-disks in M with boundary meeting B in every slope if and only if there is an embedded punctured-disk in M_A with boundary meeting B_A in a slope obtained from μ_A by Dehn twisting about α_A . In this case \mathcal{S} consists of every slope in B . From Part 2(c), if the previous situation does not hold, then

possibly the only embedded punctured-disk in M meeting B is A and there are no embedded punctured-disks in M_A meeting B_A . In this case \mathcal{S} consists of precisely the slope α . Finally, we have the possibility of an embedded punctured-disk in M with boundary meeting B in slope γ if and only if there is an embedded punctured-disk in M_A with boundary meeting B_A in a slope obtained from γ_A by Dehn twisting about α_A . In this case, the slopes in \mathcal{S} are generated by Dehn twists about α_A of slopes in \mathcal{S}_A .

Again, to find the slopes in \mathcal{S} we need to have the slopes in \mathcal{S}_A , which begins with our having the manifold M_A given to us by a triangulation. As above we find this step described in the Remark following the proof of Theorem 5.6.

The only remaining situation is that no fundamental surface in (M, \mathcal{T}) is an essential disk or an essential annulus with its boundaries in distinct components of ∂M .

In this situation, if there is an embedded, essential punctured-disk, say P , in M with its boundary in B and its punctures in $\partial M \setminus B$, then

$$P = \sum n_k F_k + \sum m_j K_j + \sum l_i A_i,$$

where $\chi(F_k) < 0$, K_j is a torus or Klein bottle, and A_i is either an annulus, with its boundary components in the same component of ∂M , or a Möbius band. Thus we have a situation for the use of an average length estimate. In this case, we can apply Proposition 5.2 where we do not restrict the slope γ and we have $b_\gamma = 1$. The average length estimate gives us that there is a constant C depending only on M and the triangulation \mathcal{T} so that any planar surface we are interested in must have a boundary shorter than $3C$.

Hence, there must be a short puncture (length less than $3C$) for all embedded punctured-disk in M having boundary B and punctures in $\partial M \setminus B$.

Let $\{\alpha_1, \dots, \alpha_N\}$ denote the slopes in the components of $\partial M \setminus B$ such that $L(\alpha_i) \leq C$. If there is any punctured-disk embedded in M with boundary in B and punctures in $\partial M \setminus B$, at least the set of punctures on some boundary must have slope one of the α_i , $1 \leq i \leq N$.

Let $(M(\alpha_i), \mathcal{T}(\alpha_i)), \dots, (M(\alpha_N), \mathcal{T}(\alpha_N))$ be the triangulated Dehn fillings of (M, \mathcal{T}) along the slopes $\alpha_1, \dots, \alpha_N$. Each $M(\alpha_j)$ is a link-manifold having n boundary components and B is a component of $\partial M(\alpha_j)$ for all j . By our induction hypothesis, and for each j , $1 \leq j \leq N$, there is a constructible set of slopes, which we denote $\mathcal{S}(\alpha_j)$, in B that satisfy conditions (i) and (ii) of the theorem. In this case $\mathcal{S} = \cup \mathcal{S}(\alpha_j)$. \square

Even though there are no embedded, essential annuli in M , an $M(\alpha_j)$ can have an embedded, essential annulus between distinct boundary components and possibly one with a component of its boundary in B . If this were the case, then $\mathcal{S}(\alpha_j)$ could have infinitely many slopes along a line, leading to \mathcal{S} having infinitely many slopes along a line. But since M , itself, has no embedded, essential annulus with boundary in distinct components of ∂M such a line of slopes in ∂M is *not* obtained by Dehn twisting about an essential annulus in M .

While every slope in \mathcal{S} is the boundary of an embedded punctured-disk in M , there may be quite different punctured-disks with the same slope for their boundary. Our methods do not find all the possible punctured-disks.

6. THE WORD PROBLEM

As mentioned earlier, this work was initiated by a study of singular normal surfaces in an attempt to solve the Word Problem for fundamental groups of 3-manifolds. In the Introduction, we gave the Word Problem for 3-manifolds as a decision problem for knot-manifolds; we then gave the Word Problem for finitely presented groups as an analogous decision problem for link-manifolds.

We begin this section with the classical version of the Word Problem for 3-manifold groups and show that it is equivalent to the version given in the Introduction.

WORD PROBLEM (closed 3-manifolds). *Given a closed 3-manifold M and a loop L in M . Decide if L is contractible in M .*

It is quite straight forward that this version is equivalent to the version that is in the Introduction; however, there is an interesting point about the version from the Introduction. Namely, we again have the issue of having an infinity of longitudes from which we want to know if one bounds a suitable punctured-disk. We address this for singular punctured-disks in Lemma 6.1 below.

We repeat here for convenience the version given in the Introduction.

WORD PROBLEM (closed 3-manifolds). *Given a knot-manifold M and a meridian on ∂M . Decide if a longitude bounds a (possibly) singular punctured-disk in M with punctures meridians.*

We first show (briefly) why the two versions are equivalent. Let $\eta(L)$ denote a small regular neighborhood of L in M . Let $M_L = M \setminus \overset{\circ}{\eta}(L)$; then M_L is a knot-manifold and we designate the unique slope on ∂M_L that bounds a disk in $\eta(L)$ as the meridian slope, μ (it is a meridian in the classical sense).

If L bounds a singular disk, say D , in M , we can set $P = D \cap M_L$ and we have a singular punctured-disk in M_L with boundary a longitude on ∂M_L and punctures meridians. Conversely, if a longitude in ∂M_L bounds a singular punctured-disk in M_L and punctures meridians, then L bounds a singular disk in M . Hence, the answer to one version of the Word Problem for 3-manifolds is yes if and only if the answer to the other version is yes.

The issue about the infinity of longitudinal slopes is easier to resolve in the singular case than in our earlier considerations regarding embedded punctured-disks.

6.1. Lemma. *Given a knot-manifold M and a meridian slope on its boundary. If any longitude on ∂M bounds a singular punctured-disk in M with punctures meridians, then all longitudes on ∂M bound a singular punctured-disk in M with punctures meridians.*

Proof. Consider Figure 8. Let μ denote the meridional slope on ∂M . Now, using the notation from Figure 8, we have that a longitude on ∂M bounds a singular punctured-disk with punctures meridians in M if and only if every longitude on ∂M , considered as a component of ∂M^μ (the manifold obtained from M by Dehn drilling along μ), bounds a singular punctured-disk in M^μ with punctures meridians on ∂M or on B^μ (the other boundary component of M^μ).

So, suppose some longitude on ∂M bounds a singular punctured-disk in M with punctures meridians. Let γ be any longitude on ∂M . Then γ bounds a singular

punctured-disk, say P_γ^μ , in M^μ with punctures meridians on ∂M or on B^μ . In fact, in this case there must be precisely one puncture with boundary on B^μ and its slope is a Dehn twist of the meridian μ^* about the longitude λ^* on B^μ ; and so has slope $\mu^* + k\lambda^*$ on B^μ for some integer k . Again, see Figure 8. However, the slope $\mu^* + k\lambda^*$ on B^μ bounds a singular punctured-disk in the solid torus $\eta(\mu_0)$ having $|k|$ punctures in B^μ with slope λ^* . The latter are all isotopic to μ on ∂M . Thus there is a singular punctured-disk D_k in M with boundary $\mu^* + k\lambda^*$ on B^μ and $|k|$ punctures in ∂M , each having slope μ . The punctured disk $P_k = P_k^\mu \cup D_k$ is a singular punctured-disk with boundary γ and punctures meridians in ∂M .

To change boundary slope, we need to change the number of punctures. But if there is a singular punctured-disk in M with boundary a longitude and punctures meridians, then there is such a *singular* punctured-disk in M for *every* longitudinal slope. \square

The classical statement of the Word Problem for finitely presented groups typically has the following form.

WORD PROBLEM (finitely presented groups). *Suppose the group G is given by the finite presentation $G = \langle X : R \rangle$. Given a word w in the symbols of X decide if $w = 1$ in G .*

In the Introduction we gave the following decision problem for link-manifolds as equivalent to the Word Problem for finitely presented groups.

WORD PROBLEM (finitely presented groups). *Given a link-manifold M , a component B of ∂M , and a meridian slope on B . Decide if there is a (possibly) singular punctured-disk in M with boundary slope a longitude in B and punctures in $\partial M \setminus B$ or meridians on B .*

We have the following construction. Let $G = \langle X : R \rangle$ be a finite presentation of the group G , where $X = \{x_1, \dots, x_n\}$ and $R = \{r_1, \dots, r_K\}$, r_j a word in the symbols of X . Let $H_n = n(S^2 \times S^1) = (S^2 \times S^1) \# \dots \# (S^2 \times S^1)$, where the right hand side has n terms. H_n is the closed handlebody of rank n ; its fundamental group is free of rank n and we label a set of free generators by x_1, \dots, x_n .

The relations r_1, \dots, r_K of R can be represented by a collection of pairwise disjoint embedded loops in H_n ; we shall denote these loops also by r_1, \dots, r_K in H_n . Let $\eta(r_j), j = 1, \dots, K$, be pairwise disjoint small tubes about r_1, \dots, r_K , respectively. Let $M_G = H_n \setminus \bigcup_{j=1}^K \overset{\circ}{\eta}(r_j)$ and denote the boundary components of M_G by $B_j = \partial \eta(r_j)$.

Now, let v_1, \dots, v_K be K distinct points and let $C_j = v_j * B_j$ be the cone on B_j with cone point v_j . Finally, let $\hat{M}_G = M_G \bigcup_{j=1}^K C_j$.

The fundamental group of \hat{M}_G is the group G .

Notice we can triangulate \hat{M}_G getting an ideal triangulation of the interior of the compact link-manifold M_G , $\hat{M}_G \setminus \{v_1, \dots, v_K\} = \overset{\circ}{M}_G$; we call the vertices v_1, \dots, v_K ideal vertices.

Now, suppose w is a word in the symbols of X . Then w can be represented by an embedded loop, also denoted w , in \hat{M}_G missing the vertices v_1, \dots, v_K . Hence, w can be represented by a loop in M_G that is equivalent to w in \hat{M}_G . We shall continue to call this loop w as a loop in M_G . Let $\eta(w)$ be a small regular neighborhood of w

in M_G and set $M = M_G \setminus \overset{\circ}{\eta}(w)$. Let $B = \partial\eta(w)$. Then M is a link-manifold with a distinguished boundary component B and a natural meridional slope, say μ .

6.2. Lemma. *The word $w = 1$ in G if and only if the loop w is contractible in \hat{M}_G if and only if for the meridional slope μ on B there is a singular punctured-disk in M with boundary a longitude in B and punctures meridional in B or on $\partial M \setminus B$.*

We conclude with a curious set of relations. If we write Condition E as $E(1)$, in the case of one boundary component, then $E(1)$ is the classical Knot-Problem; namely, is a given knot the unknot. If we write $S(1)$ for the Word Problem for 3-manifolds, then $S(1)$ is the singular version of the embedded version $E(1)$. So, the Word Problem for 3-manifolds is the singular version of the Knot Problem.

Now, write $E(n)$ for the link-manifold version of Condition E with n boundary components and call it the Link Problem. Similarly, write $S(n)$ for the 3-manifold version of the Word Problem for finitely presented groups having $n - 1$ relations, $n > 1$. Then the Word Problem for finitely presented groups is the Link Problem for 3-manifolds. A decision problem for 3-manifolds without a general solution.

We note that in using embedded normal surface theory, we employed induction to go from $E(1)$ to the general solution $E(n)$; however, following the announced solution of the Geometrization Conjecture, we have $S(1)$ solvable; so, there is no way to go from $S(1)$ to $S(n)$, leaving us with believing that singular normal surface theory can not be built as a direct analogue to the embedded theory. Of course, there are many other reasons to draw this conclusion.

REFERENCES

- [1] A. Bart and M. Scharleman. Least weight injective surfaces are fundamental. *Topology Appl.*, 69(3):251–264, 1996.
- [2] R. H. Bing. Some aspects of the Topology of 3-Manifolds Related to the Poincaré Conjecture. In *Lectures on Modern Mathematics, Vol. II*, pages 92–128. John Wiley & Sons, New York, 1964.
- [3] Wolfgang Haken. *Theorie der Normalflächen: Ein Isotopiekriterium für der Kreisknoten. Acta Math.*, 105:245–375, 1961.
- [4] A. Hatcher. On the boundary curves of incompressible surfaces. *Pacific J. Math.*, 99:373–377, 1982.
- [5] G. Hemion. *The classification of knots and 3-dimensional spaces*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.
- [6] W. Jaco and U. Oertel. An algorithm to decide if a 3-manifold is a Haken manifold. *Topology*, 23(2):195–209, 1984.
- [7] W. Jaco and L. Reeves. Cartography of projective solution space. (in preparation), 2006.
- [8] W. Jaco and H. Rubinstein. PL equivariant surgery and invariant decompositions of 3-manifolds. *Adv. in Math*, 73:149–191, 1989.
- [9] W. Jaco and J. H. Rubinstein. 0-efficient triangulations of 3-manifolds. *J. Diff. Geom.*, 65:61–168, 2003.
- [10] W. Jaco and J. H. Rubinstein. 1-efficient triangulations of 3-manifolds. 2006. (in preparation).
- [11] W. Jaco and E. Sedgwick. Decision problems in the space of Dehn fillings. *Topology*, 42:845–906, 2003.
- [12] W. Jaco and J. Tollefson. Algorithms for the complete decomposition of a closed 3-manifold. *Illionis J. Math*, 39(3):358–406, 1995.
- [13] William Jaco and J. H. Rubinstein. Blow-ups of ideal triangulations. (in preparation).
- [14] William Jaco and J. Hyam Rubinstein. Layered triangulations of three-manifolds. *ArXiv, math.GT/0603601*, pages 1–97, 2006.
- [15] J. H. Rubinstein. Polyhedral minimal surfaces, Heegaard splittings and decision problems for 3-dimensional manifolds. In *Geometric Topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 1–20. Amer. Math. Soc., Providence, RI, 1997.

- [16] H. Schubert. *Bestimmung der Primfaktorzerlegung von Verkettungen*. *Math. Zeitschr.*, 76:116–148, 1961.
- [17] A. Thompson. Thin position and the recognition problem for S^3 . *Math. Res. Lett.*, 1(5):613–630, 1994.
- [18] J. Tollefson. Isotopy classes of incompressible surfaces in irreducible 3-manifolds. *Osaka J. Math.*, 32(4):1087–1111, 1995.
- [19] F. Waldhausen. The word problem in fundamental groups of sufficiently large irreducible 3-manifolds. *Ann. of Math. (2)*, 88:272–280, 1968.

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078
E-mail address: `jaco@math.okstate.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, PARKVILLE, VIC
3052, AUSTRALIA
E-mail address: `rubin@maths.unimelb.edu.au`

SCHOOL OF CTI, DEPAUL UNIVERSITY, 243 S. WABASH AVE, CHICAGO, IL 60604
E-mail address: `esedgwick@cti.depaul.edu`